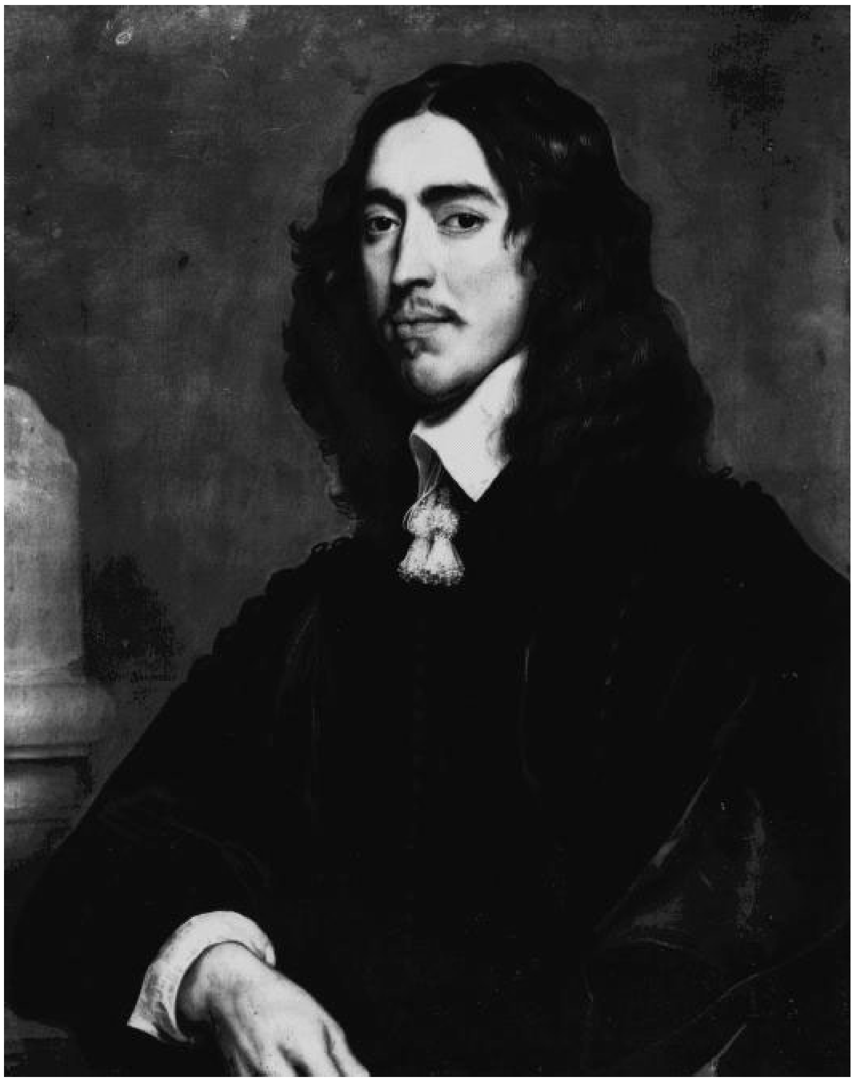
Jan de Witt's
Elementa Curvarum Linearum – Liber Secundus

Text, Translation, Introduction and Commentary by Albert W. Grootendorst

Posthumously edited by
Jan Aarts, Reinie Erné, and Miente Bakker

With 55 illustrations



Jan de Witt (1625–1672)

Portrait by Adriaen Hanneman (1601–1671)
Museum Boijmans Van Beuningen, Rotterdam

Preface by the editors

The first textbook on analytic geometry was written in Latin by the Dutch statesman and mathematician Jan de Witt. It is entitled *Elementa Curvarum Linearum*, Elements of curves, and consists of two volumes. The first volume, *liber primus*, presents the geometric generation of the conics. The second volume, *liber secundus*, deals with the classifications of quadratic curves. The second volume forms the core of the *Elementa Curvarum Linearum*, while the first volume only serves as an introduction. The basis for this textbook, which was first published in 1659, lies in the ideas that Descartes developed in his book *Géométrie* (1637).

Albert Grootendorst prepared the English translation of the first volume, *liber primus*, which was published by Springer in 2000. In addition to the parallel presentation of the Latin text and its English translation, this includes a general introduction, a summary with a complete survey of the theorems (without proofs), annotations, and two appendices. This edition is partly based on the Dutch translation by Grootendorst, which was published in 1997 by the *Centrum Wiskunde & Informatica* in Amsterdam. The Dutch translation of the second volume, *liber secundus*, by Grootendorst, was published in 2003 by the *Centrum Wiskunde & Informatica*. This book has the same format as both the Dutch and English translation of the first volume: in addition to the text and its trans-lation, it includes a general introduction, a summary of the theorems, annotations, and an appendix.

Grootendorst received much help from Miente Bakker in preparing the publication of both the Dutch and English translations. Due to his premature death in December 2004, Grootendorst was unable to complete the English edition of the second volume. At the time of his death, Grootendorst had almost completed the translation of the Latin text into English. His plan was to also translate the additional material of the Dutch edition into English. Jan Aarts completed the translation of the Latin text into English. Reinie Erné translated the additional material from the Dutch edition into English. Miente Bakker did the editorial work including preparing the index.

The Latin text that is used for this edition of the *liber secundus* is taken from the second edition of 1863 by the publisher Blaeu in Amsterdam; we are indebted to K.F. van Eijk, treasurer of the library of the Delft University of Technology, for providing access to it. Many thanks go to Tobias Baanders of the *Centrum Wiskunde & Informatica* for the additional figures illustrating the introduction, summary and annotations.

In concluding we give the last paragraph of Grootendorst's Preface of the Dutch edition of the *liber secundus*, adapted to the present situation:

This marks the completion of the English translation of the *Elementa Curvarum Linearum,* the magnum opus of the Dutch statesman Jan de Witt who was one of the greatest mathematicians of the 17th century and who might have been the greatest, had he not been distracted by so many state affairs. To quote Christiaan Huygens:

> *Nullam aeque saeculum geometrarum ferax fuisse arbitro, inter quos vir ille, si negotiis minus distringeretur vel principem locum obtinere posset.* [*In my view no century has been so rich in mathematicians, amongst whom this man* (*J. de Witt*) *might have taken the first place, had he been less distracted by state affairs.*]

We hope that this translation makes the work of the great scientist Jan de Witt accessible to a broad group of today's mathematicians.

Amsterdam, June 2009

The editors¸ Jan Aarts, Reinie Erné, and Miente Bakker

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Introduction

1. The second part of Jan de Witt´s *Elementa Curvarum Linearum* is the essence of the whole work. The first part was merely a necessary preparation to *Liber Secundus*, which is referred to in the correspondence and in Part I as the *tractatus* (or *compositio*) *locorum planorum et solidorum*.

This last text was delivered to Van Schooten at the beginning of 1658. In a letter to Jan de Witt dated 8 February 1658, Van Schooten expressed his great appreciation for this work and promised to study it carefully and to help in any way he could with the preparation for publication.

On 6 October 1658, Jan de Witt received the results of Van Schooten's efforts, with an accompanying letter in which he wrote that

> *... so veel 't mij doenlijck geweest is, accuraet (heeft) naergesien…* [*... I have checked (it) as carefully as I could...*]

In particular, the following passage is important:

> *Hebbe in het uytschryven de voorszeide calculatie op monsieur Des Cartes manier gestelt en op eenige weynige plaetsen de woorden wat verandert, om doorgaens, so veel 't mogelijck was, overal de tael sijnder geometrie, die nu by meest alle de fraeyste verstanden de allerbekendste is, te gebruycken…*[*In my exposition, I have written the aforementioned computation in the manner of Mr. Des Cartes and have changed the words in a few places in order to use, as much as possible, the language of his geometry, which is now the most current among almost all distinguished minds...*]

Jan de Witt immediately replied with a letter of October 8, in which he expressed his gratitude, but added that he had no time to spend on this work. Nevertheless, he emphasized that this text

> *Niet anders en mach voor de dach comen dan voorhenen gaende eene corte verhandelinge van de nature ende proprieteyten der cromme liniën.* [*Must not be published without being preceded by a brief treatise on the nature and properties of curved lines.*]

Apparently he had already finished this brief treatise, because he joined it to the letter, with the request to

> *... insgelijkx eens te doorsien ende de faulten ... te verbeteren* [*... look it over and correct any mistakes...*]

In the final publication this treatise became *Liber Primus*; it gives a mechanical description of the known conics as plane curves, *absque ulla solida consideratione*, that is, without any spatial considerations.

The style of *Liber Primus* is clearly different from that of *Liber Secundus*; indeed, in the first part the method of Descartes is not used at all, all calculations are done according to the geometric algebra of Euclid (as explained in Note [3.30]; see also [26]).

The definitions given there serve as the basis for *Liber Secundus*, whose core can be described as the characterization of the conics by means of equations in two variables x and y which can be seen as the coordinates (which are line segments) of the points on the curves, and the deduction of the conics' properties from this, all using the analytic method.

This results in a tightly ordered enumeration of *Theoremata* (theorems) and *Problemata* (problems). We will come back to the structure of this work in Section 10 of this introduction. It is written more in the rigid style of the *Elements* of Euclid than in the style of the *Géométrie* of Descartes and can rightly be considered the first systematic textbook of analytic geometry in the Cartesian tradition. The first book that can be seen as a successor of this work is *Elementa Matheseos Universae* of Christian von Wolff (1679−1754).

To help understand the significance of *Liber Secundus*, we will attempt to place it in the context of its time by giving some well-known, relevant facts.

2. Whenever the origin of analytical geometry is brought up, the names of René Descartes (1596−1650) and Pierre de Fermat (1601−1665) immediately come to mind. Their sources of inspiration, however, lie in a more distant past: the Greek antiquity, with names such as Euclid (ca. 300 BC), Archimedes (287−212 BC), Apollonius of Perga (second half of the third century BC), and Pappus of Alexandria (first half of the fourth century AD). Moreover, the tools that were implicitly passed on to them, and which helped make their results possible, date mostly from the late Middle Ages and the Renaissance. Of that time only Nicole Oresme (ca. 1320−1382) and François Viète (1540−1603) are mentioned here.

3. The *Collectio* (*Συναγωγη*) of Pappus occupies a central place in the developmental history of analytic geometry. Several causes can be given for this.

This collected work consisting of eight books gives an extensive overview of the work of some thirty mathematicians, from Euclid to Pappus's contemporary Hierius, and was composed by a capable mathematician. It derives its importance not only from the discussion of works known to us, but also in particular from the sometimes concise remarks concerning writings that are now lost. This is the most important aspect of the *Collectio*; because of this it led to and provided support for the reconstruction of these lost works.

Because of the renewed interest in Greek and Roman culture, including mathematics, which flourished in the Renaissance, much attention was given to this reconstruction in the 16th and 17th century.

Important contributions were made by the Frenchman Viète with his *Apollonius Gallus* (1600); by the Dutchman Snellius with his *Apollonius Batavus* (1607/1608); by the Italian (by origin) Ghetaldi with his *Apollonius Redivivus* (1607/1613); as well as by Fermat with his reconstruction of the *Loci Plani* of Apollonius, which he presented to his friend Prade around 1630 and which was one of his sources of inspiration in his setting up of analytic geometry. Frans van Schooten Jr. also attempted to reconstruct the *Loci* of Apollonius. We find the results of this attempt in his *Excercitationum Mathematicarum Libri V* (1656/1657).

In this respect the seventh book of the *Collectio* deserves particular attention. It not only contains many theorems (more than 400) from lost work, but also a definition of the fundamental terms analysis (*αναλυσις*) and synthesis (*συνθεσις*). It is known as the treasury of analysis (*τοπος αναλυομενος*).

In the opening words of this work, dedicated to his son Hermodorus, Pappus describes analysis as a subject that is intended for those who already master the well-known *Elements* and now wish to study problem solving.

Pappus names Euclid, Apollonius, and Aristaeus the Elder (ca. 350 BC) as founders of the analytic method. Heath, however, suspects that the term analysis was already known in the school of Pythagoras (see [33]).

Diogenes Laertius (3rd century AD) attributed the term to Plato, but it is probable that Plato laid the emphasis rather on the associated synthesis. In the synthetic setting up of the *Elements* of Euclid the term analysis does not come up explicitly, but it may have played a role in the heuristics. In some manuscripts of the *Elements* it appears in a marginal comment; Heiberg assumes that this is an insertion by Hero (ca. 60 AD), which refers to the research of Theaitetus (410−368 BC) or of Eudoxus (408−355 BC).

Right after the introduction Pappus gives a definition of analysis and of synthesis, as follows:

> *Analysis is a method where one assumes that which is sought, and from this, through a series of implications, arrives at something which is agreed upon on the basis of synthesis; because in analysis, one assumes that which is sought to be known, proved, or constructed, and examines what this is a consequence of and from what this latter follows, so that by backtracking we end up with something that is already known or is part of the starting points of the theory; we call such a method analysis; it is, in a sense, a solution in reversed direction. In synthesis we work in the opposite direction: we assume the last result of the analysis to be true. Then we put the causes from analysis in their natural order, as consequences, and by putting these together we obtain the proof or the construction of that which is sought. We call this synthesis.*

Let us add two remarks. First of all a linguistic observation: the word analysis is described by Pappus as anapalin lusis *(ανα παλιν λυσις*), which is the Greek term for solution in reversed order; synthesis (*συνθεσις*) means composition.

Next let us point out the remarkable manner in which analysis is described. If we indicate the stages of the analysis with $a_1, a_2, \ldots$, where a_1 is that which is sought, then, formally, Pappus's reasoning is not $a_1 \to a_2$ but $a_1 \leftarrow a_2$, where a_2 is a sufficient condition for a_1.

Pappus distinguishes two types of analysis:

1. *Theoretic* or *zetetic* analysis (from zèteo, to search for), which concerns the truth of a statement.
2. *Problematic* or *poristic* analysis (from porizo, to provide; a porism is halfway between a problem and a theorem), which concerns the constructability or computability.

When giving the definition of these two types of analysis, Pappus further remarks that for these we must assume that which is sought to be true or possible, and then, by drawing logical conclusions, end up with something known to be true or possible or known to be untrue or impossible. In the first case we must then follow the reasoning in reverse order to conclude that the assumption was correct. With this, Pappus stresses that each step in the reasoning must be reversible. In the second case the assumption is of course incorrect.

Viète would give other meanings to these terms and distinguish a third type of analysis, the *rhetic* or *exegetic* analysis. We will come back to this later.

Yet, the *Collectio* not only derives its significance for the development of analytic geometry from the references it contains and from the attention it gives to the terms analysis and synthesis, but also in particular from problem it states, which would become known as one of the Problems of Pappus.

With modern notation we can state this problem as follows:

Given three or four straight lines $l_1, l_2, l_3 (l_4)$ in the plane, we ask for the set of points *P* whose distances from $d_1, d_2, d_3 (d_4)$ satisfy

i. $d_1 d_2 : d_3^2 = \text{constant}$ (in the case of three lines),

ii. $d_1 d_2 : d_3 d_4 = \text{constant}$ (in the case of four lines).

These distances can be taken perpendicularly, but also in a direction that is determined separately for each line l_i. It is clear that this choice does not change the nature of the problem.

A possible generalization to more than four lines is obvious and is already mentioned by Pappus himself (*Collectio* vii, 38−40; ed. Hultsch p. 680, and Y. Thomas, Part II, pp. 601−603).

In the case of five lines $l_1, l_2, \ldots, l_5$, the distances are $d_1, d_2, \ldots, d_5$.

Pappus considers the two parallelepipeds that are enclosed by d_1, d_2, d_3 and by d_4, d_5 and a randomly chosen line segment *a*. The condition on *P* is now that the ratio $d_1 d_2 d_3 : a d_4 d_5$ is constant.

For six lines the distances are $d_1, d_2, ..., d_6$. The condition is then that the ratio $d_1 d_2 d_3 : d_4 d_5 d_6$ is constant. The case of four lines is shown in Figure 1.3.1. Let us already mention that in this case the loci in question will prove to be conics.

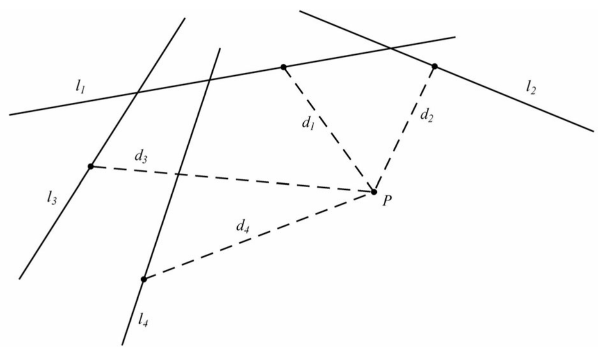

FIGURE 1.3.1

In the general case of an even number of lines, say 2*n*, the condition becomes that the ratio

$$d_1 d_2 ... d_n : d_{n+1} d_{n+2} \dots d_{2n} \text{ is constant,}$$

and in the case of an odd number of lines, say 2*n*+1, the condition becomes that the ratio

$$d_1 d_2 ... d_{n+1} : a d_{n+2} d_{n+3} \dots d_{2n+1} \text{ is constant,}$$

where *a* is again a randomly chosen line segment.

For the context of this problem we refer the reader to Section 1 of the appendix.

4. Of those who were inspired by the *Collectio* of Pappus, in particular Book VII, we first mention François Viète (1540−1603), who worked as a lawyer for the government and in his spare time was an avid mathematician.

We mentioned him above in relation to his *Apollonius Gallus*, a reconstruction of works of Apollonius, and also in relation to his views on the terms analysis and synthesis as presented by Pappus.

His most important mathematical work is considered to be *In artem analyticem isagoge*, which was published in 1591 in Tours. This work begins with a chapter on the term analysis, which already came up for discussion above. Instead of the bipartition of Pappus, he suggests his own division into three parts. For him *zetetic* analysis is the determination of an equation or proportion (with known coefficients) that the unknown quantity must satisfy. In subsequent computations this quantity is considered known. In *poristic* analysis, the correctness of a theorem is studied using this equation or proportion, while in *rhetic* or *exegetic* analysis the unknown quantity is deduced from it. In fact, Viète gives an algebraic character to the term analysis, which he describes as *doctrina bene inveniendi in mathematicis*, that is, the science of correct discovery in mathematics.

After this, he gives a consequent letter notation for arithmetic, as a result of which this arithmetic could eventually develop into an abstract algebra. Although he had predecessors in this area, such as Diophantus (ca. 250 AD) with his syncopated arithmetic notation and Bombelli (1526−1572), and Descartes improved his system, he was the first to introduce the use of vowels for unknown quantities and consonants for known quantities, both in capital letters. He also systematically used the signs + and –, but at first continued to describe equalities with words, later using the sign ~. The equality sign later introduced by Descartes, ꝏ, remained in use for a long time until it was permanently replaced by the sign = of Recorde. For powers of quantities Viète had a verbal description, which clearly betrays his geometric ideas. He had different notations for powers of known and of unknown quantities. He described the 1st, 2nd, 3rd, and 4th powers of known quantities using longitudo (or latitudo), planum, solidum, and plano-planum, and those of unknown quantities using latus (or radix), quadratum, cubus, and quadrato-quadratum. He also had analogous expressions for higher powers, up to and including the ninth power.

An example: *B A* quadratum + *C* planum *A* aequalia *D* solido,

corresponds to our: $bx^2 + c^2x = d^3$.

His geometric interpretation of products not only is reflected in his notation, but also forces him to use homogeneous formulas because, according to a condition of Aristotle, only similar quantities can and may be compared to each other. The terms of an equation are also called the *homogenea*.

It is remarkable that he speaks of powers higher than the third power in spite of the fact that these have no geometric meaning. Descartes would solve this dilemma later. Of course the innovations of Viète include more than those we mention here. These, however, are of exceptional importance: this is where the distinction arises between *logistica numerosa*, computing with explicit numbers, and *logistica speciosa*, computing with letters. As a result one could speak of equations in general terms and no longer had to rely on specific examples. Moreover, thanks to the *logistica speciosa*, the dependence of the solution on the coefficients of the equation becomes clear.

Viète used his notations in particular to solve geometric problems. To this end he transformed the problem into an algebraic equation in one variable, which he solved using his new technique, whereupon he constructed the solution, whenever possible. The ‘construction of equations’, that is, their geometric resolution, was henceforth done by means of algebra instead of the geometric and verbal techniques of antiquity (see also [26] and [27]). A well-known and much cited example is the determination of the sides of a rectangle when their ratio and the area of the rectangle are given.

Of course a construction with ruler and compass only succeeded if the equation was of degree at most two. Equations of degree three or four could be solved algebraically, but their roots could in general not be constructed with these instruments. For these cases and others Viète suggested the use of other tools. In this he did not follow the method already used by Menaechmus (ca. 350 BC), namely using common points of curves. As an example of that we give, in our notation, the method used by Menaechmus to find both geometric means x and y of a and b.

From $a : x = x : y = y : b$ follow $x^2 = ay$ and $y^2 = bx$ (and also $xy = ab$), so that $x^4 = a^2y^2 = a^2bx$ and therefore $x^3 = a^2b$. Menaechmus solved this equation by intersecting two conics he had discovered, of which he knew the geometric properties. In modern terms: he intersected the parabola determined by $x^2 = ay$ with the parabola characterized by $y^2 = bx$. Let us already mention at this point that Descartes and Fermat would continue using this method with curves of higher degree.

The above clearly shows that Viète did not get around to analytic geometry: he did not draw curves other than a straight line or a circle, and did not use coordinate systems. But the most important shortcoming is that he restricted himself to so-called 'determinate' equations, that is, equations in one variable with constant coefficients. He did not know equations in two variables, as a result of which he could not describe *loci* (*τοποι*) algebraically, where a *locus* is the set of all points whose location is determined by stated conditions.

5. The first fundamental contribution of Descartes to the foundations of analytic geometry is the creation of its own algebraic apparatus. This includes a notation that we still use today, except for the equal sign for which, as mentioned before, Descartes chose ∝. In this, though not only in this, he surpassed Viète, a fact of which he was aware: *...je commence en cela par ou Viète a finy*. [*...in this matter, I begin where Viète has left off*].

In short, the innovation of Descartes consists of defining addition, subtraction, multiplication, and division, hence also power taking and root extraction, for line segments in such a way that their results are once more line segments, so that the set of line segments is closed under these operations. As we already saw in *Liber Primus*, this was not the case for the mathematics of the Greeks: the sum and difference of line segments were indeed line segments, but the product of two line segments was a rectangle and the product of three was a rectangular parallelepiped. The product of more than three line segments was problematic (for this, see Appendix, Section 1).

Descartes presents his ideas on the first page of the *Géométrie*. It is essential for this that he introduces a fixed line segment that will serve as the unit element 1. The proportions $1 : a = b : p$ and $1 : b = q : a$ give the product $p = ab$ and the quotient $q = a/b$. Figure 1.5.1 gives the associated constructions.

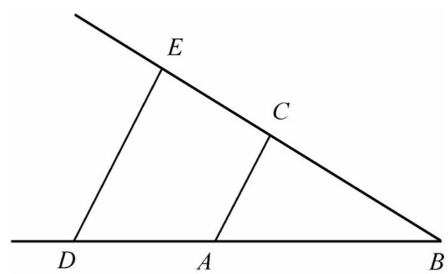

FIGURE 1.5.1

Here we have $AB = 1$ and $AC \parallel DE$, so that $1 : BD = BC : BE$, and therefore $BE = BD.BC$ and $BC = BE/BD$.

For the extraction of roots Descartes uses a well-known property of the similar triangles formed by the altitude from the right angle in a right triangle, and gives Figure 1.5.2 as example, where triangle *IFG* is similar to *HIG* and consequently $IG^2 = FG.GH$.

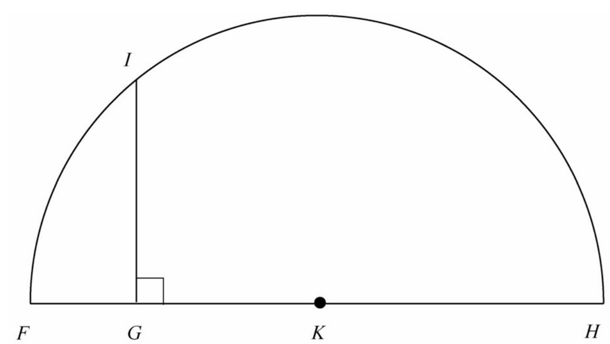

FIGURE 1.5.2

An important consequence of the introduction of the unit line segment is that Descartes can drop the condition of homogeneity. After all, all products and quotients of line segments are also line segments. Viète still required that all quantities in an equation or equality be of the same type: all line segments, all areas, or all volumes. For Descartes, however, $a^2b^2 - b$ is by definition a line segment, though he does add that, for example when extracting the cube root, this form can be written as a polynomial of degree three, that is, as $\dfrac{a^2b^2}{1} - 1^2 b.$ It is remarkable that Descartes nevertheless still speaks of the product of three line segments as *le parallelepipede composé de trois lignes* [*the parallelepiped made up of three lines*], for example in the *Géométrie* p. 336.

Right after this Descartes introduces the following notation, which is so familiar to us today: *a, b, c* ... for known line segments and ... *x, y, z* for unknown line segments. Moreover, he also introduces numerical exponents and the radical sign. We have already mentioned his variant of the equal sign. Van Schooten refers to these innovations as *the manner of Monsieur Des Cartes*.

As a first application, Descartes gives the constructive resolution of the quadratic equation $z^2 = az + b^2$. As usual he restricts himself to the positive root, dismissing the negative one as *racine fausse* (false root).

Figure 1.5.3 reflects the situation. Let the radius *NL* of the circle with center *N* be *a*/2, the length of the tangent *LM* be *b*, then we directly realize that $MO = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2}$, from which Descartes concludes right away that this is the line segment *z* in question. From the same figure he deduces that *MP* is the solution of the equation $y^2 = -ay + b^2$. For the solution of $z^2 = az - b^2$, he uses an analogous method in another figure. The equation $z^2 = -az - b^2$ without positive root is, of course, not discussed.

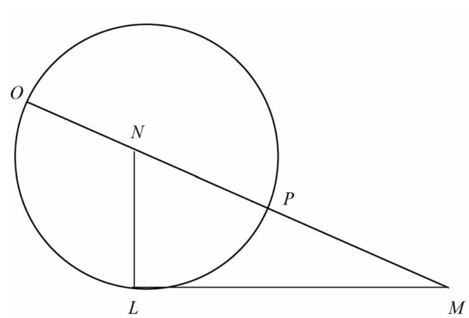

FIGURE 1.5.3

After these algebraic preparations, Descartes proceeds with his basic method for analytic geometry, which he demonstrates by means of Pappus's Problem mentioned above.

This problem was presented to him in 1631 by Jacobus Golius (1596−1667), professor of mathematics and Arabic in Leiden. Within several weeks Descartes sent a letter to Golius with his solution, which he also incorporated into the first book of the *Géométrie* (pp. 309−314 and 324−335). His treatment of this problem was the first application of the method that forms the basis of analytic geometry. Moreover, the result inspired him, among other things, in his views on what he called 'admissible curves'.

Prior to his solution Descartes discussed the place of this problem in antiquity, including a number of pointed jabs at the Greek mathematicians in general, and Pappus in particular.

He then announces his solution without much enthusiasm (*Géométrie*, p. 309):

> *En sorte que ie pense auoir entierement satisfait a ceque Pappus nous dit auoir esté cherché en cecy par les anciens & ie tascheray d'en mettre la demonstration en peu de mots, car il m'ennuie desia d'en tant escrir. [So that I think that I have fully accomplished that which, according to Pappus, the ancients wanted and I will try to give the proof in few words, as it already annoys me to write so much about it.]*

To clarify we first give Descartes's solution of Pappus's problem for four lines. We stay close to the original proof, including the notations. The associated Figure 1.5.4 is also borrowed from the *Géométrie*.

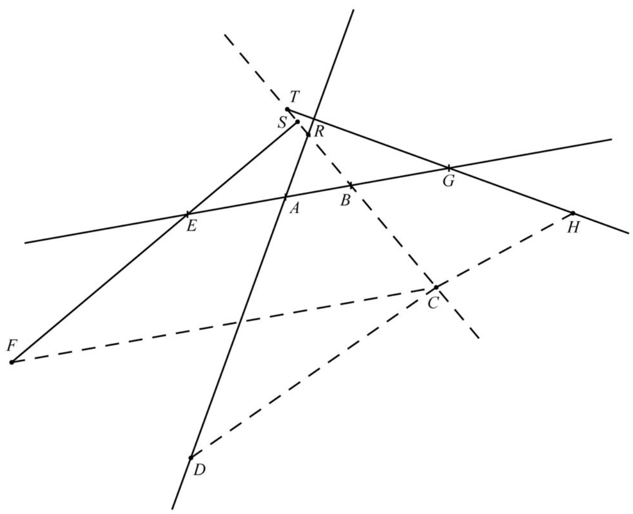

FIGURE 1.5.4

Let *AB*, *AD*, *EF*, and *GH* be four given lines in the plane. To each line corresponds a fixed direction. These directions are represented in the figure by a dotted line from a point *C*: *CB*, *CD*, *CF*, and *CH*. We want to find the position of the point *C* for which the proportion *CB.CD : CF.CH* has a given value, which Descartes chooses to be 1. We are thus looking for the equation of a plane curve that is defined geometrically. Both Fermat and Jan de Witt proceed in a different manner: they begin with a given equation and examine which curve it describes.

The method of Descartes consists of choosing the line through *A* and *B* as abscissa-axis, with *A* as origin, and the direction conjugate to *AB* as direction of the ordinate-axis, in this case from *B* to *C*. Thus the point *C* has abscissa *AB* and ordinate *BC*. The first distance we want, *CB*, is therefore equal to y.

The angles at *A* are fixed, as are the angles at *B*, so that the angles of triangle *ABR* are known, and therefore also the ratios of the sides. Descartes sets

$AB : BR = z : b$, which gives $BR = \frac{bx}{z}$, with known b and z. In our figure we therefore have $CR = y + \frac{bx}{z}$.

The angles of triangle *DCR* are also known, hence again also the ratios of the sides. Descartes sets $CR : CD = z : c$ with the same z as before. With *CR* as above, this give the second distance *CD*:

$$CD = \frac{c.CR}{z} = \frac{cy}{z} + \frac{bcx}{z^2}.$$

Descartes proceeds in an analogous manner. First he sets the known distance *AE* in triangle *SBE* equal to k, so that $EB = k + x$. For the known ratio of the sides *BE* and *BS* he writes $BE : BS = z : d$, again with the same z, whence

$$BS = \frac{d.BE}{z} = \frac{d(k+x)}{z},$$

so that

$$CS = BC + BS = y + BS = \frac{zy + dk + dx}{z}.$$

Next Descartes sets the known ratio $CS : CF$ equal to $z : e$, which gives the following result for *CF*, the third distance we were looking for:

$$CF = \frac{ezy + dek + dex}{z^2}.$$

He continues in the same manner: he sets AG = l (hence $BG = l - x$), $BG : BT = z : f$, $TC : CH = z : g$, and finally finds the fourth distance, *CH*:

$$CH = \frac{gzy + fgl - fgx}{z^2}.$$

The first conclusion Descartes draws is that, regardless of the number of given lines, the sought distances are linear forms in x and y. Of course he does not use this expression, but describes the look of such a form. For the cases where one or more lines are parallel to *AB* or to *BC*, he makes an exception to his definition of linear form, as either the term in x or the term in y is missing.

His second conclusion is that when multiplying a number of these distances, the degree of a term in x and y is never higher than the number of distances involved in the product. The locus in question is therefore represented by an equation in the two variables x and y. If one of these is given a certain value, then the other one must satisfy the resulting (determinate) equation.

He considers explicitly the case of five or less distances. The condition $d_1 d_2 d_3 = a d_4 d_5$, where a is a given line segment, leads to an equation in x and y, in which y has power at most three, but x has power at most two because the first distance, d_1, is equal to y. Repeatedly choosing random values for y gives quadratic equations in x whose roots can be constructed with ruler and compass following the method mentioned before. Descartes thus gives a point-by-point construction of the locus we are looking for.

This method can also be applied to more than five lines if a number of them are parallel to *AB* or to *AC*, where, of course, *AB* and *AC* are assumed to be concurrent. In that case it is possible that so many terms x, respectively y, cancel out that the resulting equation in x, respectively y, is of degree one or two, and that the root x, respectively y, can be constructed with ruler and compass by giving y, respectively x, a certain value.

If all lines are parallel, then five lines already pose a problem. In that case the equation can be formulated so that y does not occur; the equation in x is then of degree three. It was assumed that its roots could not be constructed with only ruler and compass; P.L. Wantzel gave the first proof of this in 1837. However, in Book II of the *Géométrie*, Descartes gives a solution of this equation using conics.

For at most nine lines that are not all parallel, the method of Descartes gives an equation in which x has degree at most four because the first distance is equal to y. Further on, in Book II, Descartes shows that in this case a solution can be given by intersecting conics. Analogously, for at most thirteen lines we obtain an equation of degree at most six in x. Descartes later solves this using a 'higher' curve, namely the 'Trident' or 'Parabola of Descartes'. We will come back to this curve in Section 3 of the appendix.

At this point Descartes interrupts his treatment of Pappus's problem to insert an overview of his classification of plane curves. He postpones his closer elaboration of Pappus's problem, which culminates in his conclusion that in the case of three or four lines the solution is a conic, to pp. 324–335 of the *Géométrie*, after which he treats the problem for five lines on pp. 335–341. In Section 2 of the appendix we will discuss this elaboration on plane curves more closely, and in Section 3 we will give Descartes' solution of Pappus's problem for five lines. We will now first continue Descartes' treatment of Pappus's problem for four lines.

Above we have already deduced values for *CB*, *CD*, *CF*, and *CH* (see also Figure 1.5.4). The condition *CB.CF* = *CD.CH* then gives

$$y^2(ez^3 - cgz^2) = y(cfglz - dekz^2 - dez^2x - cfgzx + bcgzx) + bcgflx - bcfgx^2.$$

If $ez^3 - cgz^2$ is negative, we multiply both sides by –1. Descartes only considers the values of y for which C lies inside the angle *DAG*. After introducing suitable new coefficients m and n, which depend on b, c, d, e, f, and l, Descartes reduces the equation to the form

$$y^2 = 2my - \frac{2nxy}{z} + \frac{bcfglx - bcfglx^2}{ez^3 - cgz^2}.$$

The roots of this equation are

$$y = m - \frac{nx}{z} \pm \sqrt{\left(m^2 - \frac{2mnx}{z} + \frac{n^2x^2}{z^2} + \frac{bcfglx - bcfglx^2}{ez^3 - cgz^2}\right)},$$

of which Descartes only considers the one with the plus sign. For simplification Descartes also introduces the quantities *o* and *p*, which are again dependent on the aforementioned coefficients, giving

$$y = m - \frac{nx}{z} + \sqrt{m^2 + ox - \frac{px^2}{m}}$$

for the root. Here *m*, *n*, *z*, *o*, and *p* are known quantities.

Descartes first remarks that the locus is a line if the expression under the radical sign is zero or a perfect square. Next he remarks straight off that in the other cases the locus is one of the three conics or a circle. For the construction of the parabola as a solution Descartes refers to the corresponding problem in the first book of the *Conica* of Apollonius. In the remaining cases he gives, without further explanation, the characteristic quantities for the curve (center, latus rectum, symmetry axes, vertices,...) in terms of the coefficients of the equation above. For the actual construction of the curves from this he also refers to Apollonius, after which he shows that the curves constructed this way coincide with the curves that appear as solutions for Pappus's problem.

Of course many cases can be distinguished with respect to the mutual position of the points that arise in the course of the construction, depending on the parameters of the equation. Descartes treats these in detail.

Finally, he gives a numerical example (see once more Figure 1.5.4). In this example the following values hold:

$EA = 3$; $AG = 5$; $AB = BR$; $BS = BE/2$; $GB = BT$; $CD = 3CR/2$;
$CF = 2CS$;
$CH = 2CT/3$; $\angle ABR = 60^\circ$.

The condition $CB.CF = CD.CH$ then leads to the equation

$$y^2 = 2y - xy + 5x - x^2.$$

The locus turns out to be a circle, which Descartes shows meticulously by means of his previous considerations.

With this Descartes concludes his treatment of Pappus's problem for four lines. In Section 3 of the appendix we will discuss the solution of the problem for five lines.

6. As noted before, Fermat was inspired by the work of Apollonius. Around the end of 1635 he completed a reconstruction of two lost works of Apollonius, the *Loci Plani*, which, as the title says, deal with plane loci, that is, lines and circles. This led him to write his first and fundamental contribution to analytic geometry, entitled *Ad Locos Planos et Solidos Isagoge* (*Introduction to Plane and Solid Loci*), a study consisting of only eight pages, followed by a three-page appendix.

Right from the beginning his presentation is clearly different from that of Descartes. A first example of this is his notation, which he took from Viète, even though the printed version of 1679 includes numerical exponents as used by Descartes. His setting up is also different. We have already seen that Descartes took Pappus's Problem for four lines as a starting point and established an equation for the sought locus, which he submitted to a precise examination with as final aim the construction, that is, the constructive solution, of equations.

Fermat went to work in the opposite manner; roughly speaking, he started with an equation in two variables and, using the properties of the conics from antiquity, he examined which curve it represented. This way he approached the problem of the loci in its most general form. Indeed, his reproach to the ancient mathematicians was that they had not tackled this problem in a general enough setting.

Fermat was the first to realize that an equation in two variables represents a curve. He states this insight in the following historical sentence:

> *Quoties in ultima aequalitate duae quantitates ignotae reperiuntur, fit locus loco, & terminus alterius ex illis describit lineam rectam, aut curvam, linea recta unica & simplex est, curva infinita, circulus, parabole, hyperbole, ellipsis, &c.* [*Whenever the final equation has two unknown quantities, the locus has a fixed position and the extremity of one of the two unknowns describes a straight or curved line; the straight line corresponds to only one type and is simple; there are infinitely many types of curved lines: a circle, a parabola, an ellipse, etc... .*]

The most important novelty is that in contrast to his predecessors, Fermat does not limit himself to determinate equations, that is, one equation in one variable that must be resolved, or a system with as many equations as variables, but considers indeterminate equations in two variables.

He represents the two unknown quantities in the equations as line segments in a plane. He measures off the first of the two along a fixed half-line, starting at its origin. He then sets the second one in upward direction from the origin of the first variable under a fixed, often right, angle with the first half-line (see Figure 1.6.1).

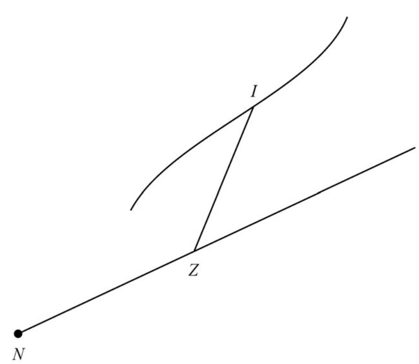

FIGURE 1.6.1

Thus Fermat uses a coordinate system with one axis. The y-axis does not occur; it comes up for the first time in a posthumous publication from 1730 by C. Rabuel (1669−1728). From now on we will speak of the abscissa and ordinate instead of the first and second variable.

The curve in question is then generated by the extremity of the ordinate for varying abscissa. Fermat calls this extremity the *terminus localis*. Generally he only considers that part of the curve that lies in what we would call the first quadrant. We will see that Jan de Witt follows this method.

The *Isagoge* of Fermat has one central theorem:

> *...modo neutra quantitatum ignotarum quadratum praetergrediatur, locus erit planus aut solidus, ut ex dicendis clarum fiet*. [*... on condition that none of the two variables occurs in a higher power than two, the locus will be plane or solid, as will become clear from what follows.*]

Here again solid locus stands for parabola, hyperbola or ellipse.

This theorem is proved by means of seven typical examples, where Fermat initially assumes the equations to be reduced (*ultima aequatio*); he had learned this reduction from the work of Viète.

In fact, with his examples Fermat studies all standard linear and quadratic equations. In this he is much more systematic than Descartes. He moreover also gives examples of translations of the axis and an example of what we would call a rotation of the x-axis. His writing contains the core of analytic geometry, though in a text that is incomplete and often impenetrable. Compared to this, the work of Jan de Witt, which builds on this and on the *Géométrie*, is an oasis of system, lucidity, and thoroughness.

In the following overview our current notation replaces the notation of Fermat. Where Fermat writes A and E for the variables, we write x and y, while we represent constants by small letters. An example: we write

$$B^2 - 2A^2 = 2AE + E^2 \text{, as}$$

$$b^2 - 2x^2 = 2xy + y^2.$$

We will also use parentheses. There were unknown to Fermat; they are of course missing from his verbal description, as a result of which the text must be read very carefully.

The equations treated by Fermat are the following:

i. $dx = by$ as an example of a straight line

ii. $xy = c^2$ as an example of a hyperbola

To this he adds an example of a translation of an axis:

$$d^2 + xy = rx + sy.$$

Using words, he describes this equation in a form that we would write as

$$(x - s)(r - y) = d^2 - rs,$$

where he remarks that we now have the same form as above if we view $x–s$ and $r–y$ as 'successors' of x and y. He does not speak explicitly of a new abscissa-axis.

iii. $x^2 = y^2$; $x^2 : y^2$ constant; and $(x^2 + y^2) : y^2 =$ constant
as examples of pairs of lines

iv. $x^2 = dy$ and $y^2 = dx$ as examples of a parabola

Next he reduces

$b^2 - x^2 = dy$ to the form $x^2 = d(r - y)$, where $dr = b^2$, and
remarks that this is the previous case if we view *r–y* as the successor of *y*.

v. $b^2 - x^2 = y^2$ as an example of a circle, at least if 'the angle' is a right angle

He then reduces $b^2 - 2dx - x^2 = y^2 + 2ry$ to $p^2 - x^2 = y^2$, where *x* and *y* have replaced *x* + *d* and *y* + *r* and where $p^2 = b^2 + d^2 + r^2$.

vi. $(b^2 - x^2) : y^2 =$ constant as an example of an ellipse

He adds to this that if the constant has value 1 and 'the angle' is a right angle, then this is a circle, but if the angle is not right, it is indeed a true ellipse.

vii $(x^2 - y^2) : y^2 =$ constant as an example of a hyperbola

The method Fermat uses for his proofs is essentially the same as that of Jan de Witt. Using the coefficients that occur in the equation, he describes, with words, the line or curve in question and shows that the abscissa and ordinate of an arbitrary point on it satisfy the given equation. The converse, the compositio or synthesis, is often missing or is disposed of as obvious: *est facilis compositio*. This corresponds to the proof that any point whose abscissa and ordinate satisfy the equation lies on the curve.

In his proof Fermat of course needs a characteristic property of the curve that is being discussed. For this he calls on one of the properties that Apollonius gave for the different conics, which Fermat assumes known by his readers. Thus for the ellipse he uses a property that we will need later on, which the reader can find in Note [3.5] of the translation. The corresponding figure is included here as Figure 1.6.2. In this *CAG* is the major axis of an ellipse with center *A*, and *ED* lies in the conjugate direction. The characteristic property states that the point *D* lies on the ellipse if and only if the ratio $DE^2 : CE.EG$ is constant.

As a simple example of the method used by Fermat we choose the equation $x^2 = dy$. In this case Fermat chooses an abscissa-axis with origin *N* and an ordinate-axis in the conjugate direction, here *ZI* (see Figure 1.6.3). Next he describes, with words, a parabola with vertex *N* whose symmetry axis lies in the direction of the ordinate-axis, with conjugate axis in the direction of the abscissa-axis and latus rectum *d*. If *PI* is drawn parallel to *NZ*, then if the parabola goes through *I*, based on the characteristic for a parabola given by Apollonius, we have $PI^2 = d.PN$.

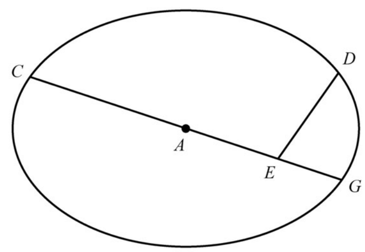

FIGURE 1.6.2

Here $PI = NZ = x$ and $NP = ZI = y$, so that a point on the curve satisfies $x^2 = dy$, which is precisely the equation of the curve that we started out with.

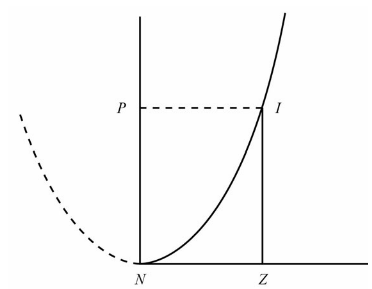

FIGURE 1.6.3

Right away Fermat remarks that, conversely, a point I satisfying $x^2 = dy$ lies on the parabola. Once more the *compositio* is missing. In the example of the circle he is clearer on the nature of the point I ; there he states explicitly that I lies on the circle in question and shows that the coordinates x and y satisfy $b^2 - x^2 = y^2$.

In addition to the examples of translations of axes mentioned above, Fermat also gives an example of a rotation of an axis, though without using this term. He announces this problem with the words:

> *Difficillima omnium aequalitatum est quando ita miscentur* A^2 *&* E^2 *ut nihilominus homogenea ab A in E afficiantur una cum datis &.*
>
> [*The most difficult of all equations is that where* x^2 *and* y^2 *occur in such a way that there are also terms with xy and constants...*]

His example is the equation

$$B^2 - 2A^2 = 2AE + E^2$$

for us $b^2 - 2x^2 = 2xy + y^2$, that is, $b^2 - x^2 = (x + y)^2$.

We will follow Fermat's solution closely, though in general using our own notation.

Fermat first chooses an abscissa-axis with origin *N* and with ordinate direction perpendicular to it (see Figure 1.6.4). *V* is an arbitrary point whose abscissa *NZ* (= *x*) and ordinate *ZV* (= *y*) satisfy the equation mentioned above. We are interested in the locus of *V* for variable *Z*.

To find this Fermat describes a circle with center *N* and radius *b* that meets the perpendicular through *Z* at *I* and the abscissa-axis at *M*.

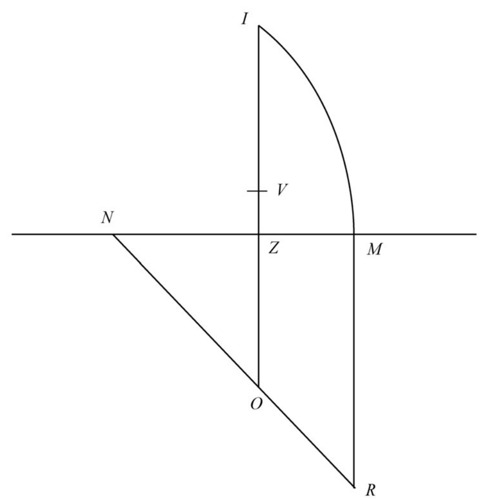

FIGURE 1.6.4

From the figure it becomes clear that

$$NM^2 - NZ^2 = ZI^2 = (ZV + VI)^2 \quad \text{(i)}$$

and from the given equation follows

$$NM^2 - NZ^2 = (ZV + NZ)^2,$$

so that $VI = NZ \ (= x)$.

Then Fermat draws the line segment *MR* parallel to *IZ* and equal to *NM*. Let *O* be the intersection point of *NR* and the extension of *IZ*. Clearly *NZ*=*ZO* ; as we have already seen that *NZ* = *VI*, this gives *OV* = *ZI*. From (i) then follows

$$NM^2 - NZ^2 = OV^2. \quad \text{(ii)}$$

Fermat now remarks that the ratios $NM^2 : NR^2$ and $NZ^2 : NO^2$ are ‘given’, without mentioning that they are equal, which is essential here. He does not use that their common value is equal to 1:2.

From $NM^2 : NR^2 = NZ^2 : NO^2 \quad = \text{constant},$

follows $(NM^2 - NZ^2) : (NR^2 - NO^2) = \text{constant}.$

Together with (ii) this gives $OV^2 : (NR^2 - NO^2) = \text{constant}.$ (iii)

From this and from the fact that the line *NR* and the angle *NOZ* are fixed he immediately concludes that the variable point *V* lies on an ellipse. Apparently he calls upon the characteristic of an ellipse mentioned above.

To check the correctness of his statement, we choose *NOR* as new abscissa-axis and the direction of *OZ* as the associated ordinate direction. The new abscissa *NO* and the new ordinate *OV* of *V* satisfy

$$OV^2 : (NR + NO) \cdot OR = \text{constant}.$$

Based on the characteristic property mentioned above, *V* lies on an ellipse with center *N*, symmetry axis *NR*, and associated conjugate axis parallel to *OV*, and hence to *RM*.

All this can be verified through a simple calculation. If we set $NO = u$ and $OV = v$, then $v^2 : (2b^2 - u^2) = 1 : 2$ follows from (iii) if we consider that $NM^2 : NR^2 = 1 : 2$.

This means that $\frac{u^2}{2b^2} + \frac{v^2}{b^2} = 1$, which is exactly the equation of an ellipse with center *N* and axes of length $2b\sqrt{2}$ and $2b$.

As $u = x\sqrt{2}$ and $v = x + y$, we have, as should be, $b^2 - x^2 = (x + y)^2$.

Fermat notes that all cases with a mixed term *xy* can be treated with similar methods.

As crowning glory (*coronidis loco*) of this work Fermat adds the following *propositio*. Consider arbitrarily many given lines in the plane and a point with line segments drawn towards these lines under given angles, then the points for which the sum of the squares of these line segments equals a given area lie on a conic.

One can see this problem as a variant of Pappus's Problem, with sums instead of products. Fermat does not solve this problem but a simplified version of it: given two points *M* and *N*, determine the locus of the points *I* such that $IM^2 + IN^2$ has a given ratio to the area of triangle *IMN*. This locus proves to be a circle, of which he gives an equation, followed by a construction independent of this.

Not without conceit Fermat concludes his *Isagoge* with the following remark:

> *If this discovery had preceded the two books on plane loci that I recently reconstructed, then the proofs of the theorems concerning loci would certainly have been more elegant.*

7. Fermat added a three-page appendix to his *Isagoge*, entitled *APPENDIX AD ISAGOGEM TOPICAM continens solutionem problematum solidorum per locos.*

[*Appendix to the introduction to the loci, containing the solution of spatial problems by means of loci.*]
In this appendix Fermat applies his theory to solving algebraic equations in one variable, the so-called determinate equations. His technique consists of introducing a cleverly chosen second variable in order to transform the problem into determining the intersection point of two plane curves. He limits himself to equations of degree three or four and shows that conics suffice for solving these.

His first example is the equation

$$x^3 + bx^2 = c^2 b\ .$$

He introduces a new variable y by setting

$$x^3 + bx^2 = c^2 b = bxy,$$

which leads to the system

$$x^2 + bx = by \qquad \text{(i)}$$

$$c^2 = xy. \qquad \text{(ii)}$$

Using this, the variable x can be found as the abscissa of an intersection point of the parabola with equation (i) and the hyperbola with equation (ii). As Fermat only considers one 'quadrant', he finds only one intersection point. He is not interested in more anyway. Any further examination of this root, such as checking whether this root indeed satisfies the equation and whether there are more roots, such as x=0, is missing. He again dismissed these matters with the words *est facilis ab analysi ad synthesim regressus*.

Fermat notes that all cubic equations can be solved using a similar method.

The second example is of degree four: $x^4 + b^3 x + c^2 x^4 = d^4$,
which he rewrites as

$$x^4 = d^4 - b^3 x - c^2 x^2.$$

By setting both sides equal to c^2y^2 one easily sees that the solution is found by intersecting the parabola with equation $x^2 = cy$ and the circle with equation $c^2 y^2 = d^4 - b^3 x - c^2 x^2$. Both here and elsewhere in the appendix Fermat mentions explicitly that one already knew how to eliminate the term x^3 in a degree four equation from the work of Viète, so that his example has a general validity.

After this Fermat brings up an old problem: determining x, the greatest of the two geometric means of b and a, where $b > a$. He of course assumes known that this satisfies the equation $x^3 = b^2 d$. To solve this he sets $x^3 = bxy$ and $b^2 d = bxy$. The x in question is therefore determined by an intersection point of the parabola with equation $x^2 = by$ and the hyperbola with equation $xy = bd$. Fermat gives a geometric explanation for this. He also gives a solution with two parabolas. For this he replaces $x^3 = b^2 d$ by $x^4 = b^2 dx$, after which he sets $x^4 = b^2 y^2$ and $b^2 dx = b^2 y^2$, so that the solution follows from 'the' intersection point of the parabolas with equations $x^2 = by$ and $y^2 = dx$. He mentions that this method can also be found in the commentary of Eutocius (ca. 480 AD) to Archimedes.

With a reproach to Viète, Fermat gives an application of his 'elegant method' to solving the general degree four equation, namely by using the intersection of a parabola and a circle. He objects to Viète's use of a cubic equation, the resultant, when solving an equation of degree four. Fermat gives two characteristic examples:

$$x^4 = c^3x + d^4 \text{ and } x^4 = c^2x^2 - c^3d.$$

Here too he notes that since Viète one knows how to get rid of a possible cubic term, so that these two types of equations suffice.

In the first case he completes the square on the left-hand side to the form $(x^2 - b^2)^2$, where b must still be determined, so that the equation becomes

$$(x^2 - b^2)^2 = c^3x + d^4 + b^4 - 2b^2x^2.$$

He then sets both sides equal to n^2y^2 with $n^2 = 2b^2$, and chooses

$$x^2 - b^2 = ny. \qquad \text{(i)}$$

We moreover have

$$c^3x + d^4 + b^4 - 2b^2x^2 = n^2y^2, \qquad \text{(ii)}$$

so that the variable x is determined by an intersection point of a parabola and a circle with equations respectively (i) and (ii).

The second case is more complicated. As before he completes both members to

$$(x^2 - b^2)^2 = b^4 - 2b^2x^2 + c^2x^2 - c^3d,$$

and sets both equal to n^2y^2. Here b and n must still be determined. Again he chooses

$$x^2 - b^2 = ny.$$

We also have $\quad b^4 - 2b^2x^2 + c^2x^2 - c^3d = n^2y^2,$

that is, $\quad (c^2 - 2b^2)x^2 + b^4 - c^3d = n^2y^2.$

It is now clear that in order to obtain a circle, we must choose b in such a way that $2b^2 > c^2$ and n in such a way that $n^2 = 2b^2 - c^2$.

The choice $2b^2 < c^2$ would lead to a hyperbola instead of the desired circle, which is simpler for a constructive solution. Fermat disregards the choice $2b^2 = c^2$.

Finally, Fermat shows that the problem of the two geometric means can also be solved in this manner. Here is a sketch of his method. First he again replaces $x^3 = b^2d$ by $x^4 = b^2dx$ and, as above, switches over to

$$(x^2 - b^2)^2 = b^4 + b^2dx - 2b^2x^2 = n^2y^2,$$

where $n^2 = 2b^2$, so that x is the abscissa of the intersection point of the parabola with equation

$$x^2 - b^2 = ny$$

and the circle with equation

$$b^4 + b^2dx - n^2x^2 = n^2y^2 \text{, that is,}$$

$$b^2 + dx - 2x^2 = 2y^2.$$

His conclusion is: whoever sees this will try in vain to solve the problem of the mesolabium, the trisection of the angle and similar problems with the help of plane curves, that is, using lines and circles. A mesolabium is an instrument for constructing the geometric means of two line segments, invented by Eratosthenes (ca. 276 − ca. 195 BC). See [65].

8. The destiny of Fermat's *Isagoge* differed in many respects from that of the *Géométrie* of Descartes.

Fermat lived isolated as Royal Counselor and Commissioner of appeals of the parliament of Toulouse and later rose through the ranks, but never ventured far from the city. He never even visited Paris. In addition to his administrative functions he practiced mathematics intensively, though he continued to consider himself an outsider. His contacts with the mathematical world were mainly through letters, among which a prominent place was taken in by the correspondence with members of a Parisian group of mathematicians led by Etienne Pascal, the father of Blaise. Père Marin Mersenne managed the extensive correspondence of this group and determined who was the right person for the various subjects.

Through this group, Fermat made his *Isagoge* (with Appendix) known to the world, and it is also thus that Descartes and Frans van Schooten Jr. obtained copies of it. Fermat, however, did not authorize the publication of his *Isagoge*, something which applied to virtually all of his work. Let us note that already prior to 1650, the brothers Elsevier had plans to publish the work of Fermat, but these were never realized. Only in 1679, fourteen years after his death, did his son Clément-Samuel provide a first printed version of his mathematical works under the title *Varia Opera Mathematica D. Petri de Fermat, Senatoris Tolosani*. However, at that point his work had already been surpassed by the *Géométrie* of Descartes and the work of his followers.

9. This *Géométrie*, which appeared directly in print in 1637 through Jan Maire, a bookseller in Leiden, was initially not appreciated by everyone. The reason for this was not only the inaccessible style of this work and the French language in which it was written, but also the fact that many continued to prefer the synthetic method of the Greeks. It was therefore important that there be mathematicians who ensured the diffusion of the ideas laid down in the *Géométrie*.

Among them Frans van Schooten Jr. deserves a special mention. He made the *Géométrie* accessible to all scholars through his translation of it into Latin, the

Lingua Franca of the scientific world, and through his lucid explanations and clear drawings. For the history of this work the reader is referred to the introduction to the translation of *Liber Primus*.

In France, G.P. de Roberval (1602−1675) spent much time on introducing the method of Descartes. Although an important work of his concerning the formulation of equations of loci and the resolution of equations using the intersection of plane curves only appeared after his death, he presumably introduced his students to the analytic geometry of Descartes during his lectures at the Collège de France.

In 1639, the *Notae Breves*, a commentary on the *Géométrie* written by Florimond Debeaune (1601−1652), came out. In this, the *Géométrie* is followed painstakingly, commented on exhaustively, and completed in a systematic manner, in particular where the classification of quadratic equations in two variables is concerned. Van Schooten Jr. included these *Notae Breves* in all editions of his *Geometria à Renato Descartes, etc*. From the second edition on, this work also included a posthumous treatise of Debeaune on algebraic equations.

Philippe de la Hire (1640−1718) was one of the French who later gave overviews and explanations of the *Géométrie*. After two dissertations in synthetic style concerning conics, he published, in 1679, a work in three parts of which the first part gives a planimetric treatment of conics as plane curves defined by their focal properties. The second part, *Les Lieux Géométriques*, is written in the style of *Liber Secundus* of Jan de Witt, but it also contains an introduction to analytic geometry in three or more dimensions. De la Hire was the first to give a simple example of a surface as locus, represented by a quadratic equation in three variables. Fermat and Descartes had only alluded to this possibility. The first systematic treatment of analytic geometry in dimension three was by Antoine Parent (1666−1716), who submitted a paper on this to the Académie des Sciences in 1700. The third part of the work of de la Hire treats the constructive resolution of algebraic equations by intersecting plane curves.

It was John Wallis (1616−1703) who propagated the ideas of Descartes in Great Britain, though not through a translation of or commentary on the *Géométrie*, but through an original work, the *Tractatus de sectionibus conicis*, published in 1655.[1] In our introduction to *Liber Primus* we have already mentioned that he defined conics by their equations. Of course these equations did not appear out of nowhere: they were inspired by the symptomata of Apollonius. In another respect Wallis also advanced the algebraification: for him coordinates were no longer line segments, but numbers. He also introduced negative abscissa and ordinates. Furthermore, in his *Arithmetica Infinitorum*, also from 1655, he linked analytic geometry and infinitesimal methods.

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[1] Editors note. One of the referees has pointed out that it can be argued that Wallis's work was almost entirely independent of Descartes's work. We refer to the English translation of *Arithmetica Infinitorum*, *The arithmetic of infinitesimals* by Jacqueline A Stedall, Sources & Studies in the History of Mathematics & the Physical Sciences, 2004.

Nevertheless, the ideas of Wallis did not catch on with all English mathematicians. Some felt more at home with the synthetic methods from antiquity; for example Newton's teacher, Isaac Barrow (1630−1677), was strongly opposed to this new method. This in contrast to Newton, who valued the *Géométrie* greatly. In continental Europe Wallis also made enemies, partly due to his claiming certain priorities. According to him, the *Géométrie* was based on the *Artis analyticae praxis* from 1631 by Thomas Harriot (1560−1621) and the *Lieux Géométriques* of de la Hire was a plagiarism of *De sectionibus conicis* of Wallis.

Of course the work of Jan de Witt has frequently been compared to *De sectionibus conicis*; we already mentioned this in the introduction to the translation of *Liber Primus*. Based on what we have said earlier, De Witt cannot have been influenced by Wallis; after all, the latter defined conics by their equations while De Witt based himself on a kinematic manner of generation and used this to show which equation corresponded to which curve. Moreover, for Wallis coordinates are numbers while for de Witt they are still line segments. One might say that de Witt is more conservative in his approach than Wallis. Finally, the first draft of De Witt's work was ready in 1649, six years before the publication of Wallis's *De sectionibus conicis*.

Let us also name Wallis's *Treatise of Algebra, both Historical and Practical*, published in 1685. In it he considered the constructive resolution of equations. It also includes the famous 'Wallis conocuneus' (conical wedge, a figure with a circular base like a cone, but having a ridge or edge instead of the apex).

Finally, let us mention Jacques Ozonam (1640−1717), who in 1687 wrote a treatise, ordered following the gradually accepted lay-out: first the geometry of conics, then loci and finally constructions of roots of algebraic equations.

By that time, however, the interest in analytic geometry was reduced, at least temporarily, by the rise of differential and integral calculus: Leibniz with his *Nova Methodus* of 1684 [44], a journal article of only six pages, and Newton, whose fundamental contribution to this new subject had been circulating amongst colleagues in manuscript form since 1665 before coming out in print in 1704.

10. To conclude we will briefly present the structure of *Liber Secundus* and sketch Jan de Witt's method. The reader will find a detailed overview of the contents in the summary.

The core of the book comes down to the following. First Jan de Witt gives equations in x and y for the straight line and the conics, in standard form with respect to a coordinate system chosen by him, which we will come back to later. These correspond to the following well-known forms:

i. $y = ax + b,\ x = ay + b$ (line),

ii. $y^2 = ax + b,\ x^2 = ay + b$ (parabola),

iii. $a^2x^2 - b^2y^2 = 1,\ a^2y^2 - b^2x^2 = 1,\ xy = c = 1$ (hyperbola),

iv. $a^2x^2 + b^2y^2 = 1$ (ellipse).

The restriction to positive coefficients and the homogeneity condition force him to distinguish more cases and to give other forms, which formally differ from these. These can be found in the summary.

He then shows that these standard forms indeed represent the curves in question. That is to say, the coordinates of every point on such a curve satisfy the corresponding equation. The converse, that every point of which the coordinates satisfy the equation indeed lies on the corresponding curve, is seldom shown.

After a short treatment of the straight line (which is hardly considered by others), all attention is on the conics. Three statements, which Jan de Witt announces as *Regula Universalis* (Universal Rule), are central.

The first concerns the parabola, and states that every quadratic equation in two variables x and y of which only one occurs as a square can be reduced to one of the forms in ii, that is, to one of the variants he gives. He also tells how this can be done: by splitting off a perfect square. He does not prove this in its full generality, but illustrates the method by means of thirteen well-chosen examples, which all correspond to rotations and to parallel translations of the abscissa-axis.

For the second *Regula Universalis* Jan de Witt assumes that the equation that is being studied can also have terms with x^2, y^2, xy, x, and y, and that it does not represent a parabola. Such an equation, he says, can be reduced to one of the forms in iii and iv. Again he shows how to do this. It is now a matter not only of splitting off a perfect square, but also of a technique that corresponds to 'removing the brackets', though he does not use brackets. For example, for the expression $xy + ay$, he introduces the new variable $v = x + a$ and substitutes $x = v - a$ in the expression, which transforms it into vy. He demonstrates this method with four fairly general examples.

In the third *Regula Universalis* Jan de Witt considers the general quadratic equation in two variables in order to show that this can always be reduced to one of the forms stated above. For this he first repeats the standard forms in x and y stated before, which represent a straight line or a conic. Then he introduces new forms for conics by successively replacing the variables y and x in the forms given in ii − iv by z and/or v, where

$$z = y + px + q \text{ and } v = x + h$$

or

$$z = y + h \text{ and } v = x + py + q.$$

In an obvious way this gives rise to the new forms. For example the form

$$a^2x^2 + b^2y^2 = 1$$

gives rise to

$$a^2x^2 + b^2z^2 = 1,\ a^2v^2 + b^2y^2 = 1, \text{ and } a^2v^2 + b^2z^2 = 1,$$

where z and v are as above.

The form $xy = c$ gives rise to $xz = c$, $vy = c$ and $vz = c$, but in this case Jan de Witt restricts himself to $z = y + h$ and $v = x + k$.

The third *Regula Universalis* states that the general quadratic equation in two variables can be reduced to one of the forms in ii−iv or to one of the forms deduced from these as above. The summary contains an overview of these. Strictly speaking, Jan de Witt does not prove that this rule holds. He goes to work conversely, as follows. He considers each of the new forms in *z* and/or *v* and shows in detail that for all possibilities for *z* and *v* it represents a conic, which he describes in detail with words. He does not write down the curves themselves, rather their vertices, centers and axes. However, he does not mention how to reduce a general quadratic equation to such a form, saying only *Methodo jam explicata*, which means ‘in the manner that has already been explained’. Apparently he means the many explicit examples, in which different manners of reduction have been shown. For example on p. [283] he clearly shows the technique that is meant here. Consequently, the third *Regula Universalis* is not followed by any examples.

Like Fermat, Jan de Witt starts out with a given linear or quadratic equation in two variables and proves that it represents one of the curves in question, as discussed above. Like Fermat, he uses a coordinate system with a one axis. The method is described in Section 6 of this introduction (see also Figure 1.6.1). Jan de Witt sets out the coordinates in the same direction as Fermat, as a result of which he too restricts himself to the ‘first quadrant’. For this De Witt uses a fixed phrase, which he repeats time after time:

> *Let A be the immutable initial point and let us suppose that x extends indefinitely along the straight line AB, and let the given or chosen angle be equal to angle ABC.*

The word ‘indefinite’ requires an explanation. It is the same word as that used in the Latin text, *indefinite*. That word can also be read as ‘infinite’ or ‘indeterminate’. ‘Infinite’ could be confusing. The quantity *x*, like *y*, remains finite, as both represent line segments. Its length can take on arbitrary values, without restriction, but remains finite. Sometimes in the translation, the word ‘indefinite’ is replace by ‘indeterminate’ or ‘arbitrary’.

In the proof that a given equation represents a certain curve (as mentioned above), De Witt proceeds in a standard manner. Using the coefficients that occur in the equation, he describes the curve that he has in mind using words. The quantities that determine this curve, such as latus rectum, vertex, symmetry axis and such, seem to appear out of nowhere, but were anticipated by him in a clever way. He calls this part of the proof the *determinatio* or the *descriptio*. After this he takes an arbitrary point on the curve with abscissa *x* and ordinate *y*. Using the geometric characteristics that he formulated in *Liber Primus* for the straight line and each of the conics, he then shows that the coordinates of the point chosen by him indeed satisfy the given equation. From p. [260] on, he explicitly calls this part of the proof the *demonstratio*.

In general De Witt does not include the converse, the *compositio* or *synthesis*. This corresponds to the proof that every point whose coordinates satisfy the equation indeed lies on the curve that is being considered. As a result of this he misses the second branch of the hyperbola. Remarkably he pays no attention to degenerations.

Like *Liber Primus*, *Liber Secundus* is a tightly ordered text. The central elements are the *Regulae Universalis* mentioned above, supported by fourteen *Theoremata*, many *Exempla* and three *Problemata*.

A *Theorema* consists of a *propositio* possibly followed by a number of *corollaria*. In the propositio a theorem is stated and proved; the corollaria give consequences of the propositio.

A *Problema* also consists of a propositio and possible corollaria. In this case the propositio states and resolves a constructive problem; the corollaria give further properties of the constructed figure.

The propositiones are numbered consecutively throughout the whole work, whether they belong to a *Theorema* or to a *Problema*.

As in *Liber Primus*, Jan de Witt has added four types of marginal notes to the text. In three of the four cases he refers to these by means of superscript numbers in the text. In the translation these marginal notes are incorporated as footnotes. This concerns:

i. References to the *Elements* of Euclid. These have the same standard form as in *Liber Primus*. For example, ‘per 16 secti’ refers to Theorem 16 of the sixth book of the *Elements*. In the footnote in the translation this is denoted by ‘VI, 16’.
ii. References to *Elementa Curvarum Linearum* itself. For example, ‘per 1 primi hujus’ refers to Proposition 1 of *Liber Primus*. To make it easier to find these we have added the number of the corresponding page of the Latin text. The footnote then reads ‘prop. 1, Lib.I, p. [162]’. Likewise ‘per 3 Corol. 6 primi hujus’ becomes ‘Corollary 3 of Prop.6, Lib.I, p. [191]’.
iii. Technical clarifications of proofs in the text. Again an example (p. [302]): as a footnote the marginal comment ‘quippe quadr. ex *HO* aequatur *GAF* rectang. ex hypoth.’ becomes ‘because, by hypothesis, the square on *HO* is equal to the rectangle *GAF*’.

There are also marginal notes that are not referred to by a number. In general this concerns the name of a special case, such as on p. [318]: ‘Casus 1^{mus}, cùm Locus est Hyperbola’. In the translation such a marginal note has in general been incorporated into the text as a heading, here: ‘First case where the locus is a hyperbola’.

In one case (p. [305]) the comment concerns the definition of a symbol used by Jan de Witt; it can be found in Note [4.2].

2

Summary

In this summary the theorems and their corollaries (*corollaria*) are restated in our modern notation, without proofs.

The essence of the statements, however, has been preserved. For the proofs we refer to the text, the translation, and the notes. This summary aims only at giving a global survey of the contents of the book.

For an overview of the structure of *Liber Secundus* and the method followed by Jan de Witt, we refer the reader to Section 10 of the introduction.

Chapter I

In this chapter Jan de Witt examines linear equations in x and y. Here x and y are clearly interpreted as line segments, as is evident, among other things, from the wording 'the initial point of one of the quantities'. The coefficients that occur are consequently always positive.

As in *Liber Primus*, the curves that are represented by the equations are called 'loci'.

The constructions in this work involve an x-axis, but no y-axis occurs. This x-axis is introduced as follows: Jan de Witt takes a fixed point A on an arbitrary line and chooses this point as the fixed and immutable initial point of the variable x, which can extend indefinitely along this fixed line through A (always to the right). A given x is then represented by a line segment on this line, for example AE. From the endpoint of this x a line segment y is drawn 'upward' under a fixed given or chosen angle (see ED in Figure 2.1). We would say that he restricts himself to the first quadrant. As was noted in the introduction, the y-axis was only introduced towards the end of the 17th century, by Claude Rabuel (1669-1728).

Chapter I consists of the following theorems:

Theorem I. *Proposition* 1**.**
If the equation is $y = \frac{bx}{a}$, then the required locus will be a straight line.

Theorem II. *Proposition* 2.
If the equation is $y = \frac{bx}{a} + c$, then the required locus will be a straight line.

Theorem III. *Proposition* 3.
If the equation is $y = \frac{bx}{a} - c$, then the required locus will be a straight line.

Theorem IV. *Proposition* 4.
If the equation is $y = c - \frac{bx}{a}$, then the required locus will be a straight line.

.
Theorem V. *Proposition* 5.
If the equation is $y = c$, then the required locus will be a straight line.

Theorem VI. *Proposition* 6.
If the equation is $x = c$, then the required locus will be a straight line.

Chapter II

In this chapter the following equations are considered:

I.	$y^2 = ax$	or conversely	$ay = x^2$;
II.	$y^2 = ax + b^2$	or conversely	$ay + b^2 = x^2$;
III.	$y^2 = ax - b^2$	or conversely	$ay - b^2 = x^2$;
IV.	$y^2 = - ax + b^2$	or conversely	$b^2 - ay = x^2$.

First the following theorems are proven:

Theorem VII. *Proposition* 7.
If the equation is $y^2 = ax$ or conversely $ay = x^2$, then the required locus will be a parabola.

This is a direct consequence of Theorem I of *Liber Primus* (p. [162]), where this property is deduced as a characteristic (*symptoma*) of the parabola.

Theorem VIII. *Proposition* 8.
If the equation is $y^2 = ax + b^2$ or conversely $ay + b^2 = x^2$, then the required locus will be a parabola.

Theorem IX. *Proposition* 9.
If the equation is $y^2 = ax - b^2$ or conversely $ay - b^2 = x^2$, then the required locus will be a parabola.

Theorem X. *Proposition* 10.
If the equation is $y^2 = -ax + b^2$ or conversely $b^2 - ay = x^2$, then the required locus will be a parabola.

The 'converse' forms (*conversim*) in Theorems VII to X are deduced by interchanging the roles of x and y in the figures and argumentations. Here too only the parts of the curves that lie to the right of A and above the x-axis are considered.

General Rule and method of reducing all equations that result from a suitable operation (when the required locus is a parabola) to one of the four cases that have been explained in the four preceding theorems

This general Rule gives a method of reducing the equation in question, when it represents a parabola, to one of the forms of Theorem VII, VIII, IX, or X. Jan de Witt does not mention how one first determines that the equation represents a parabola.

This method amounts to splitting off a square: if in addition to the term with x^2, the terms $\pm 2ax$, $\pm 2xy$, $\pm 2axy$ also occur, then we introduce a new variable z with respectively

$$z = x \pm a, \quad z = x \pm y \quad \text{or} \quad z = x \pm ay\,.$$

Of course the sign used for z is precisely the sign of the corresponding term if it occurs on the same side of the equal sign as x^2. The same holds, mutatis mutandis, if y^2 is concerned.

The examples with which Jan de Witt illustrates his method are obviously chosen so that they represent parabolas.

Examples of the reduction of equations to the form of Theorem VII

1. Reduction of the equation

$$y^2 + 2ay = bx - a^2$$

to the form $z^2 = bx$,
where $z = y + a$.

This reduction is followed by the construction of the corresponding parabola (*determinatio*, also *descriptio*) and the proof that the coordinates of the points on it indeed satisfy the equation (*demonstratio*). Henceforth, in the interest of conciseness, we will refer to this procedure as *determinatio* and *demonstratio*.

2. Reduction of the equation

$$y^2 - 2ay = bx - a^2$$

to the form $z^2 = bx$,

where $z = y - a$.

For the rest of the proof we refer to the previous example.

3. Reduction of the equation

$$by - a^2 = x^2 + 2ax$$

to the form $by = v^2$,

where $v = x + a$.

Description of the construction of this parabola in the plane, followed by the *demonstratio*.

4. Statement that the equation

$$by - a^2 = x^2 - 2ax$$

can be treated analogously.

5. Reduction of the equation

$$y^2 + \frac{2bxy}{a} + 2cy = bx - \frac{b^2x^2}{a^2} - c^2$$

to the form $z^2 = dx$,

where $z = y + \frac{bx}{a} + c$ and $d = \frac{2bc}{a} + b$.

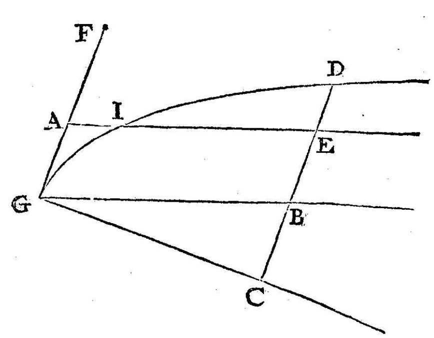

FIGURE 2.1

For the construction of this parabola (Figure 2.1), the point $G(0, -c)$ is chosen as vertex. The axis GC goes through G and lies so that $GB : BC = a : b$, where BC lies in the direction of the ordinate axis.

Here too a *demonstratio* follows, in which Jan de Witt remarks that if the curve did not meet the x-axis, the solution would also be a parabola, but would not be constructible in a 'satisfying manner' (*quod nulla tamen quaestioni satisfaciens describi possit*). By this he means that it would not lie above the x-axis. See the corresponding notes to the translation.

Note that this example was chosen very carefully.

6. The position of the curve defined by the equation

$$y^2 - \frac{2bxy}{a} - 2cy = bx - \frac{b^2x^2}{a^2} - c^2$$

is compared with that of the parabola considered in 5.

7. Reduction of the equation

$$by - \frac{b^2y^2}{a^2} - c^2 = x^2 + \frac{2bxy}{a} + 2cx$$

to the form $dy = v^2$,

where $v = x + \dfrac{by}{a} + c$ and $d = \dfrac{2bc}{a} + b$.

Note that this situation is the 'converse' of that in 5. Again a complete *determinatio* and *demonstratio* follow.

Examples of the reduction of equations to the form of Theorem VIII

1. Reduction of the equation

$$y^2 - \frac{bxy}{a} = -\frac{b^2x^2}{4a^2} + bx + d^2$$

to the form $z^2 = bx + d^2$,

where $z = y - \dfrac{bx}{2a}$.

Description (*descriptio*) of the construction of this parabola in the plane, including the proof (*demonstratio*) that the curve described this way is the parabola determined by the equation. Attention is also drawn to the 'converse' parabola, whose equation is obtained by interchanging the variables x and y.

2. Reduction of the equation

$$\frac{bcy}{a} + by - \frac{b^2y^2}{a^2} + \frac{c^2}{4} = x^2 + \frac{2byx}{a} - cx$$

to the form $by + \frac{c^2}{2} = v^2$,

where $v = x + \frac{by}{a} - \frac{c}{2}$.

Description of the construction of this parabola in the plane, including the proof that the curve described this way is the parabola determined by the equation.

Example of the reduction of equations to the form of Theorem IX

Reduction of the equation

$$y^2 + \frac{bxy}{a} - cy = ax - \frac{b^2x^2}{4a^2} - c^2$$

to the form $z^2 = dx - 3\frac{c^2}{4}$,

where $z = y + \frac{bx}{2a} - \frac{c}{2}$ and $d = a - \frac{bc}{2a}$.

Description of the construction of this parabola in the plane, including the proof that the curve described this way is the parabola determined by the equation.

Examples of the reduction of equations to the form of Theorem X

1. Reduction of the equation

$$ay - y^2 = bx$$

to the form $z^2 = \frac{a^2}{4} - bx$,

where $z = y - \frac{a}{2}$.

Description of the construction of this parabola in the plane, including the proof that the curve described this way is the parabola determined by the equation.

2. Reduction of the equation

$$\frac{b^2y^2}{a^2} + dy - c^2 = \frac{2byx}{a} - x^2$$

to the form $c^2 - dy = v^2$,

where $v = x - \frac{by}{a}$.

Description of the construction of this parabola in the plane, including the proof that the curve described this way is the parabola determined by the equation.

Another determination of the corresponding diameter and latus rectum is also given.

Problem I. *Proposition* 11**.**
Given a point and a line, determine the locus of all points in the plane passing through both that are equidistant from this point and line. Construct this locus.

The locus turns out to be a parabola. The term *focus* or umbilical point (*umbilicus*) is introduced here. The term *directrix* is not used yet.

Corollary 1. The line segment from a point *D* on a parabola to the focus is equal to the line segment from the projection of *D* on the symmetry axis to the vertex plus one fourth of the latus rectum. See Figure 2.2.

Corollary 2. The angle between the line segment from a point on the parabola to the focus and the tangent to the parabola at this point is equal to the angle between this tangent and the symmetry axis. This tangent bisects the angle between the first line segment and the line parallel to the axis and through the point on the parabola.

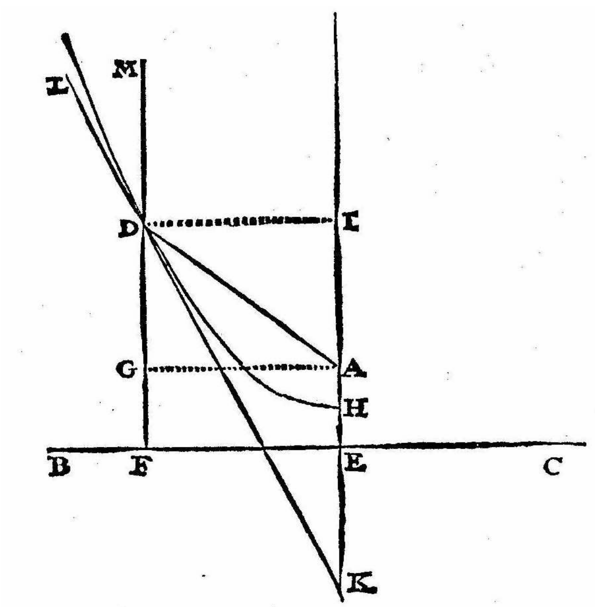

FIGURE 2.2

Chapter III

In this chapter the following equations are considered:

I. $yx = f^2$

II. $\frac{ly^2}{g} = x^2 - f^2$

III. $y^2 - f^2 = \frac{lx^2}{g}$

IV. $\frac{ly^2}{g} = f^2 - x^2$

Theorem XI. *Proposition* 12.
If the equation is $yx = f^2$, then the required locus is a hyperbola.

The proof is based on the characteristic property of a hyperbola that is deduced from the definition of a hyperbola in *Liber Primus*, Theorem III (p. [180]).

We would now say that the asymptotes are taken as 'coordinate axes'. In those days we would have said that the intersection point of the asymptotes is the origin of the abscissa-axis, which lies along an asymptote, while the angle between the asymptotes is the 'given or chosen' angle.

Theorem XII. *Proposition* 13.
If the equation is $\frac{ly^2}{g} = x^2 - f^2$, then the required locus is a hyperbola.
The proof is based on Theorem IX of *Liber Primus* (p. [196]). Using this property of the hyperbola, Jan de Witt identified 'his' hyperbola with the hyperbola of the ancient Greeks, and in particular that of Apollonius.

Theorem XIII. *Proposition* 14.
If the equation is $y^2 - f^2 = \frac{lx^2}{g}$, then the required locus is a hyperbola.
The proof and associated construction are analogous to the argumentation for Theorem XII. In fact, the roles of x and y are interchanged.

Theorem XIV. *Proposition* 15.
If the equation is $\frac{ly^2}{g} = f^2 - x^2$, then the required locus is an ellipse (or a circle).
The proof of this theorem is based directly on the characteristic property that Jan de Witt deduced in *Liber Primus*, Theorem XII (p. [205]) from his definition of the ellipse. Of course the curve is a circle if $l = g$ and the 'given or chosen' angle is a right angle.

General Rule and method of reducing all equations that result from a suitable operation (when the locus is a hyperbola or an ellipse or a circle) to one of the four cases that have been explained in the four preceding theorems

This general Rule states, without proof, that an equation of degree two that contains one or more of the terms xy, x^2, y^2, ax, and by, can be reduced to one of the forms of Theorems XI to XIV, at least if it represents a hyperbola, an ellipse, or a circle.
The method amounts to replacing the combinations

$$xy \pm ay \text{ and } xy \pm ax$$

by vy, resp. vx, where $v = x \pm a$, resp. $v = y \pm a$, and splitting off a square from the combinations

$$x^2 \pm 2\,ax, \text{ resp. } y^2 \pm 2\,ay$$

by introducing the new variables

$$v = x \pm a, \text{ resp. } v = y \pm a.$$

As Jan de Witt did not have any parentheses at his disposal, from our point of view these computations are rather long-winded. For more details see the notes to the translation of the passage in question.

Example of the reduction of equations to the form of Theorem XI

Reduction of the equation

$$yx - cx + hy = e^2$$

to the form $zv = f^2$,
where $z = y - c$, $v = x + h$, and $f^2 = e^2 - ch$.
Description of the construction of this hyperbola in the plane, including the proof that the curve described this way is the hyperbola determined by the equation.

Examples of the reduction of equations to the forms of Theorems XII and XIII

1. Reduction of the equation

$$y^2 + \frac{2bxy}{a} + 2cy = \frac{fx^2}{a} + ex + d^2.$$

The equation is first reduced to

$$\frac{a^2z^2}{fa+b^2} = v^2 - h^2 + \frac{a^2d^2 + a^2c^2}{fa+b^2},$$

where $z = y + \frac{bx}{a} + c$, $v = x + \frac{a^2e + 2abc}{2fa + 2b^2}$, and $$2h = \frac{a^2 e + 2abc}{fa + b^2}\ .$$

Then two cases are distinguished:

i. $$h^2 > \frac{a^2 d^2 + a^2 c^2}{fa + b^2}\ ;$$

ii. $$h^2 < \frac{a^2 d^2 + a^2 c^2}{fa + b^2}\ .$$

In the first case the curve turns out to be a hyperbola opened towards the x-axis. In the second case the curve turns out to be a hyperbola with the rounded side towards the x-axis.

A detailed *descriptio* with complete *demonstratio* is given for these curves.

2. Reduction of the equation
$$x^2 + 2ay = \frac{2bxy}{a}$$
to the form $$z^2 - \frac{a^6}{b^4} = \frac{a^2 v^2}{b^2}\ ,\text{ where } z = y - \frac{a^3}{b^2} \text{ and } v = x - \frac{by}{a}\ .$$
Again this includes a detailed *determinatio* and *demonstratio.*

Problem II. *Proposition* 16.
Given two points, find a third point with the property that the line segments drawn from this point to each of the two given points differ by a given distance and determine and describe the locus to which the required point belongs.

This locus turns out to be a hyperbola, as can be concluded from the equation that is obtained, which is treated in Theorem XII. This can also be verified using the definition of a hyperbola derived from the 'application problems' from Greek antiquity. The two given points are designated as foci.

Jan de Witt tacitly assumes that the solution lies in the plane.

Corollary 1.
If from a point selected at random on a hyperbola, segments are drawn to both umbilici, the longest of them will exceed the shortest by the length of the transverse axis.

Corollary 2.
If from a point selected at random on a hyperbola, straight lines are drawn to both umbilici, then the line that bisects the angle enclosed by these straight lines touches the curve at this point and conversely.

Example of the reduction of equations to the form of Theorem XIV

Reduction of the equation

$$y^2 + \frac{2bxy}{a} - 2cy = -x^2 + dx + k^2$$

to the form

$$z^2 = \frac{-a^2x^2 + b^2x^2}{a^2} + \frac{dax - 2bcx}{a} + c^2 + k^2,$$

where $z = y - c + \dfrac{bx}{a}$.

This equation is then reduced to

$$\frac{a^2z^2}{a^2 - b^2} = -v^2 + h^2 + \frac{c^2a^2 + k^2a^2}{a^2 - b^2},$$

where $2h = \dfrac{da^2 - 2bca}{a^2 - b^2}$

and $v = x - h$.

Assuming that $a^2 > b^2$, this equation is further reduced to

$$\frac{lz^2}{g} = f^2 - v^2,$$

where $\dfrac{l}{g} = \dfrac{a^2}{a^2 - b^2}$

and $f^2 = h^2 + \dfrac{c^2a^2 + k^2a^2}{a^2 - b^2}$.

The equation has now been reduced to the form of Theorem XIV and the conclusion is that it represents an ellipse or a circle. Again a detailed *descriptio* and the associated *demonstratio* are given.

The case $a^2 < b^2$ is not considered; indeed, it would not lead to an example of Theorem XIV.

Problem III. *Proposition* 17.
Given two points, find a third point with the property that the segments drawn from this point to each of the given points are, taken together, equal to a given length, and determine and describe the locus to which the required points belong.

This locus turns out to be an ellipse, as can be concluded from the equation that is obtained, which is treated in Theorem XIV. This can also be verified using the definition of an ellipse derived from the 'application problems' from Greek antiquity. Again the two given points are designated as foci.

Jan de Witt again tacitly assumes that the solution lies in the plane.

Corollary 1
Taken together, the segments drawn from an arbitrary point on an ellipse to each of the umbilici are equal to the length of the transverse axis.

Corollary 2
If one draws straight lines from an arbitrary point on an ellipse to each of the umbilici, then the exterior bisection of the resulting angle will touch the curve at the aforementioned point.

Conversely, the tangent at this point forms equal angles with the extended line segments joining this point to the umbilici.

Chapter IV

General Rule to find and determine arbitrary plane and solid loci

In this chapter a classification of the general equation of degree at most two in the undetermined quantities x and y is undertaken, and for each equation the corresponding line or curve is examined.

For the terms ‘plane’ and ‘solid’ loci, see the introduction.

Jan de Witt distinguishes the following cases:

1. $y = \frac{bx}{a}$, or $y = x$, if $a = b$.

$$y = \frac{bx}{a} \pm c, \qquad \text{or } y = c - \frac{bx}{a}.$$

He notes that one of the two quantities x or y may be missing. This remark is necessary because a and b are known *positive* quantities (line segments).

2.
$$\begin{array}{lll} y^2 = dx & & dy = x^2 \\ y^2 = dx \bullet f^2 & \text{or conversely} & dy \bullet f^2 = x^2 \\ z^2 = dx & & dy = v^2 \\ z^2 = dx \bullet f^2 & & dy \bullet f^2 = v^2. \end{array}$$

3.
$$\begin{array}{lll} y^2 = \frac{lx^2}{g} \bullet f^2 & & yx = f^2 \\ z^2 = \frac{lx^2}{g} \bullet f^2 & \text{or even} & zx = f^2 \end{array}$$

$$y^2 = \frac{lv^2}{g} \bullet f^2 \qquad\qquad yv = f^2$$

$$z^2 = \frac{lv^2}{g} \bullet f^2 \qquad\qquad zv = f^2.$$

Remarks:

1. A term of the form $A \bullet B$ represents three cases, namely, $A + B$, $A - B$, and $-A + B$. The case $-A - B$ does not occur as A and B, as well as the left-hand side, must all be positive quantities.
2. In addition to the original variables x and y, variables v and z also occur in these equations. Two cases must be distinguished:
   i. z has the form $z = y \pm c$, $z = y \pm \frac{bx}{a}$, or $z = y \pm \frac{bx}{a} \pm c$, in which case v has the form $v = x \pm h$, and therefore does not contain any term with y.
   ii. v has the form $v = x \pm c$, $v = x \pm \frac{by}{a}$, or $v = x + \frac{by}{a} \pm c$, in which case z has the form $z = y \pm h$, and therefore does not contain any term with x.

   Here a, b, c, and h are known positive quantities (line segments).
3. The general Rule states that every quadratic equation in the unknown quantities x and y can be reduced to one of the forms mentioned in 2 and 3. For the forms $zx = f^2$ and $yv = f^2$ we only allow $z = y \pm h$ and $v = x \pm k$. De Witt does not prove this rule explicitly; rather, he refers to the methods applied in the examples. In what follows he shows that the equations mentioned in 2 and 3 all represent conics. See also Section 10 of the introduction.

In order to help the reader make his way through this chapter, let us first state its the global lay-out.

The following are treated in succession:

i. From p. [306] to [307]: the straight line
ii. From p. [308] to [314], line 4 from the bottom: the parabola given by one of the equations in the first column in 2, p. [305] (p. 41 of this summary)
iii. From p. [314], line 3 from the bottom, to p. [318], line 9 from the top: the parabola given by one of the equations in the second column in 2; that is, the converse of ii, where x and y have been interchanged
iv. From p. [318], line 10 from the top, to p. [330]: the equations in the first column in 3 where the term with x^2 or v^2 has a plus sign; these equations turn out to represent hyperbolas
v. From p. [331] to [332], line 7 from the bottom: the equations in the second column in 3, which also turn out to represent hyperbolas
vi. From p. [332], line 6 from the bottom to the end of the chapter: the equations in the first column in 2 in which the term with x^2 or v^2 has a minus sign,

while obviously f^2 has a plus sign; these equations turn out to represent ellipses or circles

As in previous chapters Jan de Witt chooses an axis *AB* along which the abscissas *x* are measured out, in positive direction only, and an angle *ABE* under which the ordinates *y* are positioned. Of the curves treated here, only the part above the axis is considered.

The straight line

In case 1 on p. [305] (p. 41 in the summary) – the straight line – the following cases are distinguished and illustrated:

$$y = x, y = \frac{bx}{a}, y = \frac{bx}{a} + c, y = \frac{bx}{a} - c,\ y = c - \frac{bx}{a}.$$

First the constructions are given in the form of lists of instructions; this is followed by a meticulous proof that the abscissa and ordinate of any point on one of the constructed lines satisfy the equation in question. For an illustration, see Figure 2.3 (p. [307] of the text).

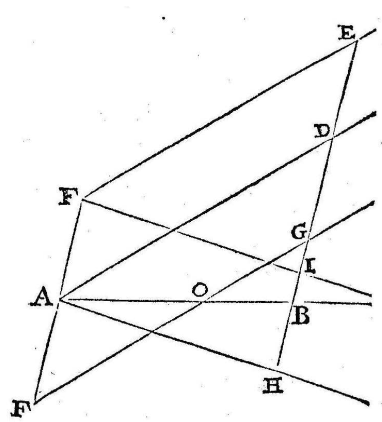

FIGURE 2.3

The parabola

In case 2 – the parabola – the left column is treated first. Nine cases are distinguished, each of which is split up into subcases. Jan de Witt begins with a list of strict instructions for the constructions. At the end of those he remarks that the proof that the constructed curves satisfy the corresponding equations is not difficult (*Quorum quidem omnium demonstratio perfacilis est*, p. [311], line 6 from the top). Only then does he begin with these proofs.

As far as the corresponding illustrations are concerned, note that Jan de Witt only gives the positions of the transverse axes with associated conjugate axes and vertices, but never draws a curve.

Moreover, for a given case he uses the same letter for the vertex in all associated subcases. For example in Case IX (p. [310] to p. [314]) we can distinguish nine subcases; in each of them the letter Q denotes the vertex of the corresponding parabola.

Again the construction of the curves characterized above by equations is given without proof, in the form of lists of instructions.

This concerns the following situations: the figure that corresponds to Cases I to IX is Figure 2.4 (p. [308]).

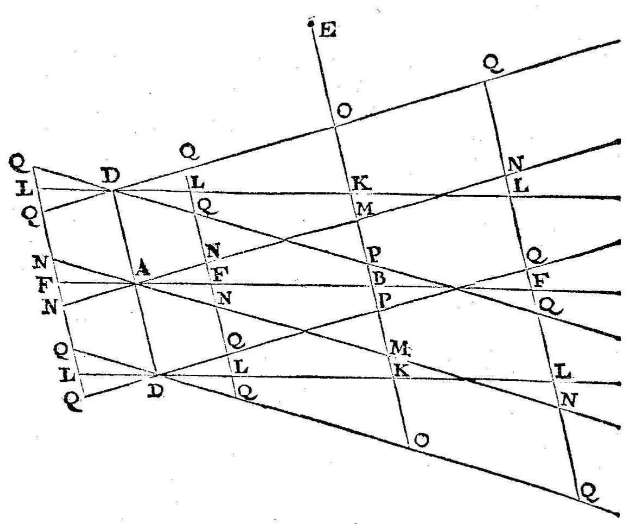

FIGURE 2.4

I. $y^2 = dx$

This equation represents a parabola with transverse axis *AB* and vertex *A*, while the ordinate-wise applied lines make an angle with this axis that is equal to the given or chosen angle *ABE*.

II. $y^2 = dx \bullet f^2$

This equation represents a parabola with transverse axis on *AB*. For the vertex *F* we have:

If $y^2 = dx + f^2$, then the vertex is $F(-\frac{f^2}{d}, 0)$.

If $y^2 = dx - f^2$, then the vertex is $F(\frac{f^2}{d}, 0)$.

If $y^2 = - dx + f^2$, then the vertex is $F(\frac{f^2}{d}, 0)$.

In the last case the parabola 'has its opening to the left'.

III. $z^2 = dx$, where $z = y \pm c$
In this case the line $y = c$ is the transverse axis if $z = y - c$, while $y = - c$ is the transverse axis if $z = y + c$; the vertex is $D(c, 0)$, resp. $D(-c, 0)$.

IV. $z^2 = dx \bullet f^2$, where $z = y \pm c$
In this case the line $y = c$ is the transverse axis if $z = y - c$, while $y = - c$ is the transverse axis if $z = y + c$; for the vertex L we have:

If $z^2 = dx + f^2$ and $z = y \pm c$, then the vertex is $L(-\frac{f^2}{d}, \mp c)$.

If $z^2 = dx - f^2$ and $z = y \pm c$, then the vertex is $L(\frac{f^2}{d}, \mp c)$.

If $z^2 = -dx + f^2$ and $z = y \pm c$, then the vertex is $L(\frac{f^2}{d}, \mp c)$, but then the parabola 'has its opening to the left'.

In Cases I to IV the latus rectum is equal to d; the conjugate axis lies in the direction of line BE.

V. $z^2 = dx$, where $z = y \pm \frac{bx}{a}$
In this case one chooses a point M on BE so that $AB : BM = a : b$. The point M lies 'above' the line AB if $z = y + \frac{bx}{a}$ and 'below' the line AB if $z = y + \frac{bx}{a}$; the support of AM is then the transverse axis, the conjugate axis lies in the direction of BE; the corresponding vertex is A.

VI. $z^2 = dx \bullet f^2$, where $z = y \pm \frac{bx}{a}$
Again one chooses AM as in Case V, but next one draws the lines FL through the points F and L (see II and IV for the definitions). These lines meet the lines AM at the points N. The

following cases are distinguished with respect to the position of the parabola:

1. $z^2 = dx + f^2$ and $z = y + \frac{bx}{a}$: the line AM with equation $y = -\frac{bx}{a}$ is the transverse axis, the corresponding vertex N has abscissa $AF = -\frac{f^2}{d}$
2. $z^2 = dx - f^2$ and $z = y + \frac{bx}{a}$: the line AM with equation $y = -\frac{bx}{a}$ is the transverse axis, the corresponding vertex N has abscissa $AF = \frac{f^2}{d}$
3. $z^2 = -dx + f^2$ and $z = y + \frac{bx}{a}$: the line AM with equation y $= -\frac{bx}{a}$ is the transverse axis, the corresponding vertex N has abscissa $AF = \frac{f^2}{d}$, but now the parabola 'has its opening to the left'

The cases $z^2 = dx + f^2$, $z^2 = dx - f^2$, and $z^2 = -dx + f^2$, where $z = y - \frac{bx}{a}$, are treated analogously.

VII. $z^2 = dx$, where $z = y + \frac{bx}{a} + c$ or $z = y - \frac{bx}{a} - c$

1. $z^2 = dx$, $z = y + \frac{bx}{a} + c$: the vertex is $D(-c, 0)$, the transverse axis $y = -\frac{bx}{a} - c$
2. $z^2 = dx$, $z = y - \frac{bx}{a} - c$: the vertex is $D(c, 0)$, the transverse axis $y = \frac{bx}{a} + c$

VIII. $z^2 = dx$, where $z = y + \frac{bx}{a} - c$ or $z = y - \frac{bx}{a} + c$

1. $z^2 = dx$, $z = y + \dfrac{bx}{a} - c$: the vertex is $D(c, 0)$, the transverse axis $y = -\dfrac{bx}{a} + c$

2. $z^2 = dx$, $z = y - \dfrac{bx}{a} - c$: the vertex is $D(-c, 0)$, the transverse axis $y = \dfrac{bx}{a} - c$

In Cases VII and VIII the conjugate axes lie parallel to *BE*.

IX. $z^2 = dx \bullet f^2$, where $z = y \pm \dfrac{bx}{a} \pm c$

1. $z^2 = dx + f^2$: the abscissa of the vertex *Q* is $-\dfrac{f^2}{d}$

2. $z^2 = dx - f^2$: the abscissa of the vertex *Q* is $\dfrac{f^2}{d}$

3. $z^2 = -dx + f^2$: the abscissa of the vertex *Q* is $\dfrac{f^2}{d}$

but now the parabola 'has its opening to the left'.

The position of the corresponding axes and vertices can be found as described in VII and VIII. In all this concerns nine subcases.

Finally we find the statement that in Cases V to IX the latus rectum *p* satisfies the proportion $e : a = d : p$, where *e* is defined by $AB : BM : AM = a : b : e$ (see Figure 2.4).

In each of these nine cases, the parabolas are described through lists of instructions that use the parameters of the given equation. After the statements above, Jan de Witt proves that these parabolas, which seem to appear out of nowhere, are indeed represented by the equations he started out with.

Jan de Witt concludes his treatment of the parabola with a short discussion of the 'converse' cases, where *x* and *y* have been interchanged, that is,

$$dy = x^2,\ dy \bullet f^2 = x^2,\ dy = v^2,\ dy \bullet f^2 = v^2,$$

where successively

$$v = x \pm c,\ v = x \pm \frac{by}{a},\ \text{or } v = x \pm \frac{by}{a} \pm c.$$

The hyperbola

In case 3 on p. [305] (p. 41 of this summary) Jan de Witt first considers the equations in the first column that represent a hyperbola, that is, whose terms with x^2 or v^2 have a plus sign; then he considers the equations in the second column that also represent a hyperbola, that is,

$$yx = f^2,\ zx = f^2,\ yv = f^2, \text{ and } zv = f^2,$$

where $z = y \pm c$ and $v = x \pm h$.

A. From p. [318] on the following modifications of the first column in 3, p. [305], are distinguished.

I. $\dfrac{lx^2}{g} = y^2 - f^2$ or $\dfrac{ly^2}{g} = x^2 - f^2$

II. $\dfrac{lx^2}{g} = z^2 - f^2$ or $\dfrac{lz^2}{g} = x^2 - f^2$, where

1. $z = y \pm c$
2. $z = y \pm \dfrac{bx}{a}$
3. $z = y \pm \dfrac{bx}{a} \pm c$

III. $\dfrac{lv^2}{g} = y^2 - f^2$ or $\dfrac{ly^2}{g} = v^2 - f^2$, where $v = x \pm h$

IV. $\dfrac{lv^2}{g} = z^2 - f^2$ or $\dfrac{lz^2}{g} = v^2 - f^2$, where $v = x \pm h$ and

1. $z = y \pm c$
2. $z = y \pm \dfrac{bx}{a}$
3. $z = y \pm \dfrac{bx}{a} \pm c$

As one can see the cases that are distinguished are those where x and y, resp. x and z, y and v, or z and v have been interchanged. Again one receives the description of such a hyperbola as a list of instructions, followed by the proof that the coordinates of the points on it satisfy the initial equation.

Here are the results (see also Figure 2.6).

I. $\frac{lx^2}{g} = y^2 - f^2$ or $\frac{ly^2}{g} = x^2 - f^2$

In the first case one chooses the transverse axis of the hyperbola along the line *AX*, through *A* and parallel to the line *BE*; in the second case one chooses the transverse axis along the line *AB*. The associated conjugate axes are parallel to respectively *AB* and *BE*. In both cases *A* is the center and 2*f* the length of the transverse diameter.

The associated latus rectum *p* is determined, that is, chosen, using the proportion $2f : p = l : g$. The length 2*d* of the conjugate diameter then follows from the definition of the latus rectum using the proportion $2f : 2d = 2d : p$.

In our further explanation and proof of the correctness of the construction we restrict ourselves to the second equation that was mentioned; Jan de Witt of course treats both equations. The point *B* on the curve is chosen as the intersection point of the curve with the line through *B* that makes the given angle with the abscissa-axis *AE*. Jan de Witt now refers to the characteristic property of the hyperbola that he mentioned in *Liber Primus* as Theorem IX, Proposition 10, p. [196], illustrated with the figure on p. [198] (here Figure 2.5). For the situation in Figure 2.5, this characteristic property implies that:

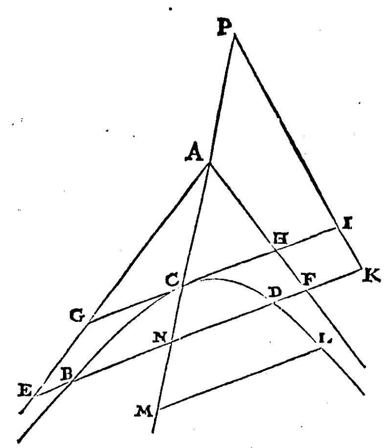

FIGURE 2.5

$$ND^2 : NC \cdot NP = GH^2 : CP^2 = CH^2 : AC^2.$$

That is, $$ND^2 : (NA - AC)(NA + AC) = CH^2 : AC^2.$$

If we now set $ND = y$, $NA = x$, $AC = f$, $CH = d$, then this property implies that

$$y^2 : (x-f)(x+f) = d^2 : f^2.$$

The ratio $d^2 : f^2$ can be determined as follows. By the choice of p we have $2f : p = l : g$ and by the definition of p we have

$$2f : 2d = 2d : p,$$

so that $d^2 : f^2 = p : 2f = g : l$

and therefore

$$\frac{ly^2}{g} = x^2 - f^2.$$

In passing let us remark that Jan de Witt treats the cases $l = g$ and $l \neq g$ separately, just as further on, in his treatment of the ellipse, where he treats the circle as a distinct case.

Again he only shows that the coordinates of the points on the constructed curve satisfy the equation, but does not search for all points whose coordinates have this property. As a consequence he only finds one branch of the hyperbola.

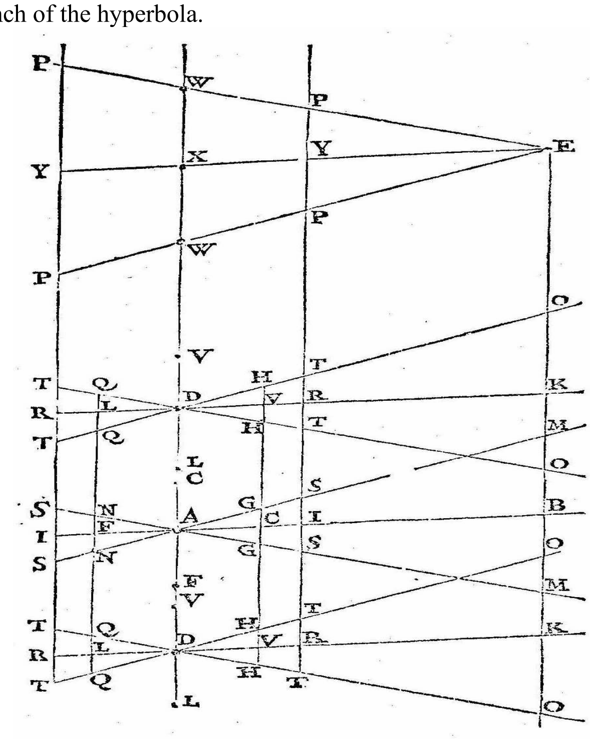

FIGURE 2.6

II.1. $\dfrac{lx^2}{g} = z^2 - f^2$ or $\dfrac{lz^2}{g} = x^2 - f^2$, where $z = y \pm c$

In this case one chooses $D(0, c)$ as center if $z = y - c$ and $D(0, -c)$ if $z = y + c$. After this the reasoning is analogous to that in I, where D takes the place of A. This now concerns hyperbolas with transverse axis $x = 0$, respectively $y = c$ if $z = y - c$ and hyperbolas with transverse axis $x = 0$, respectively $y = -c$ if $z = y + c$. The curves have been translated over a distance of $\pm c$ in the ordinate direction.

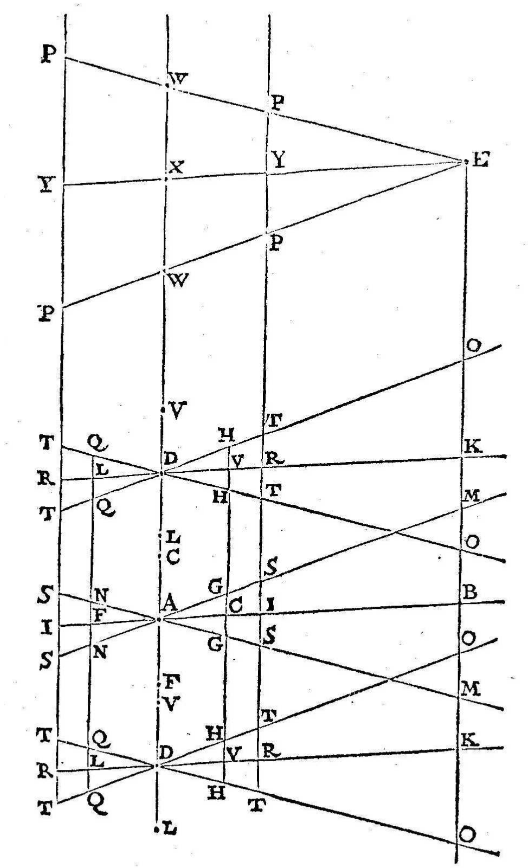

FIGURE 2.7

II.2. $\frac{lx^2}{g} = z^2 - f^2$ or $\frac{lz^2}{g} = x^2 - f^2$, where $z = y \pm \frac{bx}{a}$

In both cases one chooses the point A as center. Then one chooses the line AM with equation $y = \frac{bx}{a}$ if $z = y - \frac{bx}{a}$, and the line AM with equation $y = -\frac{bx}{a}$ if $z = y + \frac{bx}{a}$ (see Figure 2.7, which is the figure on p. [323]).

In the first case the transverse axis of the required hyperbola lies on the line AW, parallel to BE. The conjugate axis lies in the direction of AM (corresponding to $z = y + \frac{bx}{a}$ or $z = y - \frac{bx}{a}$).

In the second case the transverse axis of the required hyperbola lies on the line AM (corresponding to $z = y + \frac{bx}{a}$ or $z = y - \frac{bx}{a}$). The conjugate axis lies in the direction of AW.

In the first case the length of the transverse diameter of the hyperbola (on AW) is equal to $2f$. The corresponding latus rectum p is determined by the proportion $2f : p = a^2 l : e^2 g$, where we have

$$AB : BM : AM = a : b : e.$$

The length $2d$ of the conjugate diameter follows from the definition of the latus rectum as the third element of the proportion involving $2f$ and $2d$, that is, from

$$2f : 2d = 2d : p.$$

The ratio $d^2 : f^2$ that is so important is then equal to $\frac{e^2 g}{a^2 l}$.

In the second case the length $2m$ of the transverse diameter of the hyperbola (on AM) is equal to $\frac{2ef}{a}$. The vertices therefore lie at the intersection points of the lines AM with the lines $x = \pm a$. In this case the corresponding latus rectum p is determined by the proportion $2m : p = e^2 l : a^2 g$. The length $2d$ of the conjugate diameter again follows from the definition of the latus rectum as the third element of the proportion involving $2m$ and $2d$, that is, from $2m : 2d = 2d : p$. The ratio $d^2 : m^2$ is then equal to $\frac{a^2 g}{e^2 l}$.

This description of the required hyperbola is again followed by a proof that the coordinates of the points on the constructed curve indeed satisfy the initial equations.

II.3. $\frac{lx^2}{g} = z^2 - f^2$ or $\frac{lz^2}{g} = x^2 - f^2$, where $z = y \pm \frac{bx}{a} \pm c$

The reasoning is analogous to that in II.2, where D (0, c) takes the place of A if $z = y \pm \frac{bx}{a} - c$ and $D(0, -c)$ if $z = y \pm \frac{bx}{a} + c$. Here too the curves have been translated over a distance of $\pm c$ in the ordinate direction.

III. $\frac{lv^2}{g} = y^2 - f^2$ or $\frac{ly^2}{g} = v^2 - f^2$, where $v = x \pm h$

In this case one chooses I (h, 0) as center if $v = x - h$ and I ($-h$, 0) if $v = x + h$. The reasoning is analogous to that in I, where it now concerns hyperbolas with transverse axis on $x = h$ respectively $y = 0$ if $v = x - h$, and hyperbolas with transverse axis on $x = -h$ respectively $y = 0$ if $v = x + h$. The curves have been translated over a distance of $\pm h$ in the abscissa direction.

IV.1. $\frac{lv^2}{g} = z^2 - f^2$ or $\frac{lz^2}{g} = v^2 - f^2$, where $v = x \pm h$ and $z = y \pm c$.

In this case one chooses $R(\pm h, \pm c)$ as center, where the signs correspond to those chosen in

$v = x \pm h$ and $z = y \pm c$

The reasoning is analogous to that in I; this again concerns hyperbolas to which a parallel translation has been applied.

IV.2. $\frac{lv^2}{g} = z^2 - f^2$ or $\frac{lz^2}{g} = v^2 - f^2$, where $v = x \pm h$ and $z = y \pm \frac{bx}{a}$

In this case one chooses $S(\pm h, \pm \frac{bh}{a})$ as center, where the signs correspond to those chosen in $v = x \mp h$ and $z = y \mp \frac{bx}{a}$. The reasoning is analogous to that in II.2.

IV.3. $\frac{lv^2}{g} = z^2 - f^2$ or $\frac{lz^2}{g} = v^2 - f^2$, where $v = x \pm h$ and $z = y \pm \frac{bx}{a} \pm c$

In this case one chooses $T(\pm\ h,\ \pm\dfrac{bh}{a}\pm\ c)$ as center, where the signs correspond to those chosen in $v=\mp h$ and $z=y\ \mp\dfrac{bx}{a}\mp c$. The reasoning is analogous to that in II.3.

B. Four other cases where the required locus is a hyperbola

This concerns the equations

1. $yx=f^2$
2. $zx=f^2$
3. $yv=f^2$
4. $zv=f^2$,

where $z=y\pm h$ and $v=x\pm c$ and f, h, and c are known positive quantities (line segments).

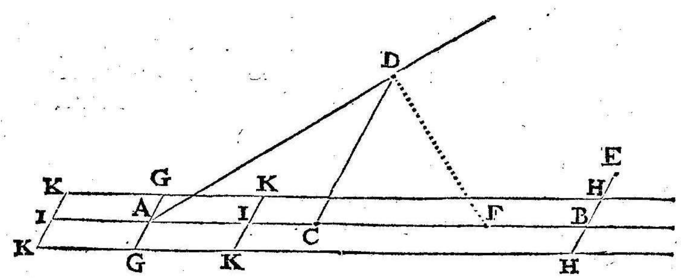

FIGURE 2.8

1. $yx=f^2$

For the construction of the required hyperbola the line segment $AC=f$ is measured out on the abscissa-axis AB (see Figure 2.8). Then the line segment $CD=f$ is set out in the ordinate direction.

The required locus will turn out to be a hyperbola with center A, with transverse axis, of length $2AD$, lying along the support of AD (hence with vertex D) and with asymptote AB.

That the coordinates of the points on this curve satisfy the equation $yx=f^2$ simply follows from the characteristic property of the hyperbola that one finds in *Liber Primus*, Theorem 3, on p. [181]. Indeed, if we let E be the intersection point of the curve with the line through B under the given angle (in the ordinate direction), then by the property mentioned above $AB.BE=AC^2$, that is, $xy=f^2$.

2. $zx = f^2$, where $z = y \pm h$
In this case one does not choose A as vertex but the point $G(0, \pm h)$, where the sign corresponds to that chosen in $z = y \mp h$. The reasoning is analogous to that in 1.

3. $yv = f^2$, where $v = x \pm c$
In this case one does not choose A as vertex but the point $I(\pm c, 0)$, where the sign corresponds to that chosen in $v = x \mp c$. The reasoning is analogous to that in 1.

4. $zv = f^2$, where $z = y \pm h$ and $v = x \pm c$
In this case one does not choose A as vertex but the point $K(\pm c, \pm h)$, where the signs correspond to those chosen in $v = x \mp c$ and $z = y \mp h$. Again the reasoning is analogous to that in 1.

The ellipse

As mentioned before, this chapter and consequently the whole work, concludes with a discussion of the remaining cases in the first column in 3 on p. [305] (pp. 40–41 of this summary), that is, those where the term with x^2 or that with v^2 has a minus sign. The term f^2 then of course has a plus sign. These equations turn out to represent ellipses or circles.
Jan de Witt does not take exactly the equations in the column in question, but starts with the formula $\frac{ly^2}{g} = f^2 - x^2$ and treats it together with a number of modifications. He distinguishes the following cases:

I. $\frac{ly^2}{g} = f^2 - x^2$

II. $\frac{lz^2}{g} = f^2 - x^2$, where

1. $z = y \pm c$
2. $z = y \pm \frac{bx}{a}$
3. $z = y \pm c \pm \frac{bx}{a}$

III. $\dfrac{ly^2}{g} = f^2 - v^2$, where $v = x \pm h$

IV. $\dfrac{lz^2}{g} = f^2 - v^2$, where $v = x \pm h$ and

1. $z = y \pm c$
2. $z = y \pm \dfrac{bx}{a}$
3. $z = y \pm c \pm \dfrac{bx}{a}$

Here are the results he obtains (see Figure 2.9).

I. $\dfrac{ly^2}{g} = f^2 - x^2$

The claim is that this concerns an ellipse that can be described as follows: The center is *A*, the transverse diameter *FAC* lies on the support of *AB* and is of length 2*f*, so that *FA* = *AC* = *f*. The conjugate diameter lies in the direction of line *BE*, and the corresponding latus rectum is determined by the condition $2f : p = l : g$.

To prove that the described curve is the required locus, Jan de Witt takes a point (*x*,*y*) on the curve and using his geometric definition of an ellipse (see *Liber Primus*, Theorem XII, p. [205]) shows that the coordinates of this point satisfy the equation in question. He also remarks that if $l = g$ and the angle *ABE* is a right angle, then the curve is a circle.

We note that once more he has not proved that he has found all points that satisfy the equation.

II.1. $\dfrac{lz^2}{g} = f^2 - x^2$, where $z = y \pm c$

This concerns an ellipse (or a circle) with center *D*(0, *c*) if $z = y - c$ and *D*(0, – *c*) if $z = y + c$.

The reasoning is analogous to that in I, where *D* takes the place of A.

II.2. $\dfrac{lz^2}{g} = f^2 - x^2$, where $z = y \pm \dfrac{bx}{a}$

Jan de Witt now constructs an ellipse as follows: The center is *A*, the transverse axis *NAG* lies on the support of *AM*, which has the following equation:

$$y = \frac{bx}{a} \quad (\text{if } z = y - \frac{bx}{a})$$

and $y = -\frac{bx}{a}$ (if $z = y + \frac{bx}{a}$).

The conjugate axis lies in the direction of line BE. The length $2m$ of the transverse diameter NG *is* $2\frac{ef}{a}$, where we again have

$AB : BM : AM = a : b : e$ (see Figure 2.9).

The corresponding latus rectum is determined by the condition

$2m : p = e^2l : a^2g.$

To prove that the described curve is the required locus, Jan de Witt takes an arbitrary point (x, y) on the curve and using his geometric definition of an ellipse shows that the coordinates of this point satisfy the equation in question.

He also remarks that if $e^2l = a^2g$ and the angle AME is a right angle, then the curve is a circle.

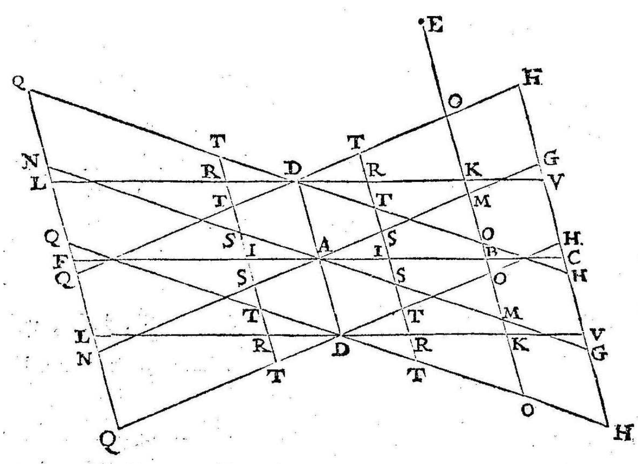

FIGURE 2.9

II.3. $\frac{lz^2}{g} = f^2 - x^2$, where $z = y \pm c \pm \frac{bx}{a}$

This concerns an ellipse (or a circle) with center D $(0, c)$ if $z = y - c \pm \frac{bx}{a}$ and D $(0, -c)$ if $z = y + c \pm \frac{bx}{a}$. The reasoning is analogous to that in II.2, where D takes the place of A.

III. $\frac{ly^2}{g} = f^2 - v^2$, where $v = x \pm h$

This concerns an ellipse with center $I(h, 0)$ if $v = x - h$ and $I(-h, 0)$ if $v = x + h$. The reasoning is analogous to that in I, where I takes the place of A.

IV.1. $\frac{lz^2}{g} = f^2 - v^2$, where $v = x \pm h$ and $z = y \pm c$

This concerns an ellipse with center $R(\pm h, \pm c)$, where the signs correspond to those chosen $v = x \mp h$ and $z = y \mp c$. The transverse diameter lies on the line R parallel to AB and has length $2f$. The corresponding latus rectum is determined by the condition that the ratio of the transverse diameter to this latus rectum is as $l : g$. The reasoning is analogous to that in II.1, where one of the four points R takes the place of A.

IV.2. $\frac{lz^2}{g} = f^2 - v^2$, where $v = x \pm h$ and $z = y \pm \frac{bx}{a}$. This concerns an ellipse whose center S is defined as the intersection point of the line $x = \pm h$ and the line $y = \pm \frac{bx}{a}$. The transverse diameter lies on the line $y = \pm \frac{bx}{a}$ and has length $\frac{2ef}{a}$. The corresponding latus rectum is determined by the condition that the ratio of the transverse diameter to this latus rectum is as $e^2 l : a^2 g$. The reasoning is analogous to that in II.2.

IV.3. $\frac{lz^2}{g} = f^2 - v^2$, where $v = x \pm h$ and $z = y \pm c \pm \frac{bx}{a}$. This concerns an ellipse whose center T is defined as the intersection point of the line $x = \pm h$ and the line through D $(\pm h, 0)$ with equation $y = \pm h \pm \frac{bx}{a}$. The transverse diameter lies on the line $y = \pm h \pm \frac{bx}{a}$ and has length $\frac{2ef}{a}$. The corresponding latus rectum is determined by the condition that the ratio of the transverse diameter to this latus rectum is as $e^2 l : a^2 g$. The rest of the reasoning is analogous to that in II.2 and II.3.